FUNCTIONS

1.

ALGEBRAIC TECHNIQUES

This chapter revises and extends the algebraic techniques that you will need for this course. These include indices, algebraic expressions, expansion, factorisation, algebraic fractions and surds.

CHAPTER OUTLINE

IN THIS CHAPTER YOU WILL:

- identify and use index rules including fractional and negative indices
- simplify algebraic expressions
- remove grouping symbols including perfect squares and the difference of 2 squares
- factorise expressions including binomials and special factors
- simplify algebraic fractions
- use algebra to substitute into formulas
- simplify and use surds including rationalising the denominator

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TERMINOLOGY

binomial: A mathematical expression consisting of 2 terms; for example, $x + 3$ and $3x - 1$

binomial product: The product of binomial expressions; for example, $(x + 3)(2x - 1)$

expression: A mathematical statement involving numbers, pronumerals and symbols; for example, $2x - 3$

factor: A whole number that divides exactly into another number. For example, 4 is a factor of 28

factorise: To write an expression as a product of its factors; that is, take out the highest common factor in an expression and place the rest in brackets. For example, $2y - 8 = 2(y - 4)$

index: The power or exponent of a number. For example, 2^3 has a base number of 2 and an index of 3. The plural of index is **indices**

power: The index or exponent of a number. For example, 2^3 has a base number of 2 and a power of 3

root: A number that when multiplied by itself a given number of times equals another number. For example, $\sqrt{25} = 5$ because $5^2 = 25$

surd: A root that can't be simplified; for example, $\sqrt{3}$

term: A part of an expression containing pronumerals and/or numbers separated by an operation such as +, −, × or ÷. For example, in $2x - 3$ the terms are $2x$ and 3

trinomial: An expression with 3 terms; for example, $3x^2 - 2x + 1$

1.01 Index laws

An **index** (or **power** or **exponent**) of a number shows how many times a number is multiplied by itself. A **root** of a number is the inverse of the power.

For example:

- $4^3 = 4 \times 4 \times 4 = 64$
- $2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$
- $\sqrt{36} = 6$ since $6^2 = 36$
- $\sqrt[3]{8} = 2$ since $2^3 = 8$
- $\sqrt[6]{64} = 2$ since $2^6 = 64$

Note: In 4^3 the 4 is called the base number and the 3 is called the index or power.

There are some general laws that simplify calculations with indices. These laws work for any m and n, including fractions and negative numbers.

Index laws

$$a^m \times a^n = a^{m+n}$$

$$a^m \div a^n = a^{m-n}$$

$$(a^m)^n = a^{mn}$$

$$(ab)^n = a^n b^n$$

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

ISBN 9780170413152

EXAMPLE 1

Simplify:

a $m^9 \times m^7 \div m^2$ **b** $(2y^4)^3$ **c** $\dfrac{(y^6)^3 \times y^{-4}}{y^5}$

Solution

a $m^9 \times m^7 \div m^2 = m^{9+7-2}$
$= m^{14}$

b $(2y^4)^3 = 2^3(y^4)^3$
$= 2^3y^{4\times 3}$
$= 8y^{12}$

c $\dfrac{(y^6)^3 \times y^{-4}}{y^5} = \dfrac{y^{18} \times y^{-4}}{y^5}$
$= \dfrac{y^{18+(-4)}}{y^5}$
$= \dfrac{y^{14}}{y^5}$
$= y^{14-5}$
$= y^9$

Exercise 1.01 Index laws

1 Evaluate without using a calculator:

a $5^3 \times 2^2$ **b** $3^4 + 8^2$ **c** $\left(\dfrac{1}{4}\right)^3$
d $\sqrt[3]{27}$ **e** $\sqrt[4]{16}$

2 Evaluate correct to 1 decimal place:

a 3.7^2 **b** $1.06^{1.5}$ **c** $2.3^{-0.2}$
d $\sqrt[3]{19}$ **e** $\sqrt[3]{34.8 - 1.2 \times 43.1}$ **f** $\dfrac{1}{\sqrt[3]{0.99 + 5.61}}$

3 Simplify:

a $a^6 \times a^9 \times a^2$ **b** $y^3 \times y^{-8} \times y^5$ **c** $a^{-1} \times a^{-3}$
d $w^{\frac{1}{2}} \times w^{\frac{1}{2}}$ **e** $x^6 \div x$ **f** $p^3 \div p^{-7}$
g $\dfrac{y^{11}}{y^5}$ **h** $(x^7)^3$ **i** $(2x^5)^2$
j $(3y^{-2})^4$ **k** $a^3 \times a^5 \div a^7$ **l** $\left(\dfrac{x^2}{y^9}\right)^5$
m $\dfrac{w^6 \times w^7}{w^3}$ **n** $\dfrac{p^2 \times (p^3)^4}{p^9}$ **o** $\dfrac{x^6 \div x^7}{x^2}$
p $\dfrac{a^2 \times (b^2)^6}{a^4 \times b^9}$ **q** $\dfrac{(x^2)^{-3} \times (y^3)^2}{x^{-1} \times y^4}$

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4 Simplify:

a $x^5 \times x^9$ **b** $a^{-1} \times a^{-6}$ **c** $\frac{m^7}{m^3}$

d $k^{13} \times k^6 \div k^9$ **e** $a^{-5} \times a^4 \times a^{-7}$ **f** $x^{\frac{2}{5}} \times x^{\frac{3}{5}}$

g $\frac{m^5 \times n^4}{m^4 \times n^2}$ **h** $\frac{p^{\frac{1}{2}} \times p^{\frac{1}{2}}}{p^2}$ **i** $(3x^{11})^2$

j $\frac{(x^4)^6}{x^3}$

5 Expand each expression and simplify where possible:

a $(pq^3)^5$ **b** $\left(\frac{a}{b}\right)^8$ **c** $\left(\frac{4a}{b^4}\right)^3$

d $(7a^5b)^2$ **e** $\frac{(2m^7)^3}{m^4}$ **f** $\frac{xy^3 \times (xy^2)^4}{xy}$

g $\frac{(2k^8)^4}{(6k^3)^3}$ **h** $(2y^5)^7 \times \frac{y^{12}}{8}$ **i** $\left(\frac{a^6 \times a^4}{a^{11}}\right)^{-3}$

j $\left(\frac{5xy^9}{x^8 \times y^3}\right)^3$

6 Evaluate a^3b^2 when $a = 2$ and $b = \frac{3}{4}$.

7 If $x = \frac{2}{3}$ and $y = \frac{1}{9}$, find the value of $\frac{x^3y^2}{xy^5}$.

8 If $a = \frac{1}{2}$, $b = \frac{1}{3}$ and $c = \frac{1}{4}$, evaluate $\frac{a^2b^3}{c^4}$ as a fraction.

9 **a** Simplify $\frac{a^{11}b^8}{a^8b^7}$.

b Hence evaluate $\frac{a^{11}b^8}{a^8b^7}$ as a fraction when $a = \frac{2}{5}$ and $b = \frac{5}{8}$.

10 **a** Simplify $\frac{p^5q^8r^4}{p^4q^6r^2}$.

b Hence evaluate $\frac{p^5q^8r^4}{p^4q^6r^2}$ as a fraction when $p = \frac{7}{8}$, $q = \frac{2}{3}$ and $r = \frac{3}{4}$.

11 Evaluate $(a^4)^3$ when $a = \left(\frac{2}{3}\right)^{\frac{1}{6}}$.

12 Evaluate $\frac{a^3b^6}{b^4}$ when $a = \frac{1}{2}$ and $b = \frac{2}{3}$.

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13 Evaluate $\dfrac{x^4 y^7}{x^5 y^5}$ when $x = \dfrac{1}{3}$ and $y = \dfrac{2}{9}$.

14 Evaluate $\dfrac{k^{-5}}{k^{-9}}$ when $k = \dfrac{1}{3}$.

15 Evaluate $\dfrac{a^4 b^6}{a^3 (b^2)^2}$ when $a = \dfrac{3}{4}$ and $b = \dfrac{1}{9}$.

16 Evaluate $\dfrac{a^6 \times b^3}{a^5 \times b^2}$ as a fraction when $a = \dfrac{1}{9}$ and $b = \dfrac{3}{4}$.

1.02 Zero and negative indices

Zero and negative indices

$$x^0 = 1$$

$$x^{-n} = \frac{1}{x^n}$$

WS Review of index laws

EXAMPLE 2

a Simplify $\left(\dfrac{ab^5c}{abc^4}\right)^0$.

b Evaluate 2^{-3}.

c Write in index form:

i $\dfrac{1}{x^2}$ **ii** $\dfrac{3}{x^5}$ **iii** $\dfrac{1}{5x}$ **iv** $\dfrac{1}{x+1}$

d Write a^{-3} without the negative index.

Solution

a $\left(\dfrac{ab^5c}{abc^4}\right)^0 = 1$

b $2^{-3} = \dfrac{1}{2^3}$

$= \dfrac{1}{8}$

c **i** $\dfrac{1}{x^2} = x^{-2}$

ii $\dfrac{3}{x^5} = 3 \times \dfrac{1}{x^5}$

$= 3x^{-5}$

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iii $\frac{1}{5x} = \frac{1}{5} \times \frac{1}{x}$

$= \frac{1}{5}x^{-1}$

iv $\frac{1}{x+1} = \frac{1}{(x+1)^1}$

$= (x+1)^{-1}$

d $a^{-3} = \frac{1}{a^3}$

Exercise 1.02 Zero and negative indices

1 Evaluate as a fraction or whole number:

a 3^{-3} **b** 4^{-1} **c** 7^{-3} **d** 10^{-4} **e** 2^{-8}

f 6^0 **g** 2^{-5} **h** 3^{-4} **i** 7^{-1} **j** 9^{-2}

k 2^{-6} **l** 3^{-2} **m** 4^0 **n** 6^{-2} **o** 5^{-3}

p 10^{-5} **q** 2^{-7} **r** 2^0 **s** 8^{-2} **t** 4^{-3}

2 Evaluate:

a 2^0 **b** $\left(\frac{1}{2}\right)^{-4}$ **c** $\left(\frac{2}{3}\right)^{-1}$ **d** $\left(\frac{5}{6}\right)^{-2}$ **e** $\left(\frac{x+2y}{3x-y}\right)^0$

f $\left(\frac{1}{5}\right)^{-3}$ **g** $\left(\frac{3}{4}\right)^{-1}$ **h** $\left(\frac{1}{7}\right)^{-2}$ **i** $\left(\frac{2}{3}\right)^{-3}$ **j** $\left(\frac{1}{2}\right)^{-5}$

k $\left(\frac{3}{7}\right)^{-1}$ **l** $\left(\frac{8}{9}\right)^0$ **m** $\left(\frac{6}{7}\right)^{-2}$ **n** $\left(\frac{9}{10}\right)^{-2}$ **o** $\left(\frac{6}{11}\right)^0$

p $\left(-\frac{1}{4}\right)^{-2}$ **q** $\left(-\frac{2}{5}\right)^{-3}$ **r** $\left(-3\frac{2}{7}\right)^{-1}$ **s** $\left(-\frac{3}{8}\right)^0$ **t** $\left(-1\frac{1}{4}\right)^{-2}$

3 Change into index form:

a $\frac{1}{m^3}$ **b** $\frac{1}{x}$ **c** $\frac{1}{p^7}$ **d** $\frac{1}{d^9}$ **e** $\frac{1}{k^5}$

f $\frac{1}{x^2}$ **g** $\frac{2}{x^4}$ **h** $\frac{3}{y^2}$ **i** $\frac{1}{2z^6}$ **j** $\frac{3}{5t^8}$

k $\frac{2}{7x}$ **l** $\frac{5}{2m^6}$ **m** $\frac{2}{3y^7}$ **n** $\frac{1}{(3x+4)^2}$ **o** $\frac{1}{(a+b)^8}$

p $\frac{1}{x-2}$ **q** $\frac{1}{(5p+1)^3}$ **r** $\frac{2}{(4t-9)^5}$ **s** $\frac{1}{4(x+1)^{11}}$ **t** $\frac{5}{9(a+3b)^7}$

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4 Write without negative indices:

a t^{-5} **b** x^{-6} **c** y^{-3} **d** n^{-8} **e** w^{-10}

f $2x^{-1}$ **g** $3m^{-4}$ **h** $5x^{-7}$ **i** $(2x)^{-3}$ **j** $(4n)^{-1}$

k $(x+1)^{-6}$ **l** $(8y+z)^{-1}$ **m** $(k-3)^{-2}$ **n** $(3x+2y)^{-9}$ **o** $\left(\frac{1}{x}\right)^{-5}$

p $\left(\frac{1}{y}\right)^{-10}$ **q** $\left(\frac{2}{p}\right)^{-1}$ **r** $\left(\frac{1}{a+b}\right)^{-2}$ **s** $\left(\frac{x+y}{x-y}\right)^{-1}$ **t** $\left(\frac{2w-z}{3x+y}\right)^{-7}$

1.03 Fractional indices

INVESTIGATION

FRACTIONAL INDICES

Consider the following examples.

$$\left(x^{\frac{1}{2}}\right)^2 = x^1 \text{ (by index laws)}$$
$$= x$$

$$\left(\sqrt{x}\right)^2 = x$$

$$\text{So } \left(x^{\frac{1}{2}}\right)^2 = \left(\sqrt{x}\right)^2$$
$$= x$$
$$\therefore \quad x^{\frac{1}{2}} = \sqrt{x}$$

Now simplify these expressions.

1 $\left(x^2\right)^{\frac{1}{2}}$ **2** $\sqrt{x^2}$ **3** $\left(x^{\frac{1}{3}}\right)^3$ **4** $\left(x^3\right)^{\frac{1}{3}}$ **5** $\left(\sqrt[3]{x}\right)^3$

6 $\sqrt[3]{x^3}$ **7** $\left(x^{\frac{1}{4}}\right)^4$ **8** $\left(x^4\right)^{\frac{1}{4}}$ **9** $\left(\sqrt[4]{x}\right)^4$ **10** $\sqrt[4]{x^4}$

Use your results to complete:

$x^{\frac{1}{n}} =$

WS Indices

WS Fractional indices and radicals

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Power of $\frac{1}{n}$

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

Proof

$$\left(a^{\frac{1}{n}}\right)^n = a \quad \text{(by index laws)}$$

$$(\sqrt[n]{a})^n = a$$

$$\therefore a^{\frac{1}{n}} = \sqrt[n]{a}$$

EXAMPLE 3

a Evaluate:

i $49^{\frac{1}{2}}$ **ii** $27^{\frac{1}{3}}$

b Write $\sqrt{3x-2}$ in index form.

c Write $(a+b)^{\frac{1}{7}}$ without fractional indices.

Solution

a **i** $49^{\frac{1}{2}} = \sqrt{49} = 7$ **ii** $27^{\frac{1}{3}} = \sqrt[3]{27} = 3$

b $\sqrt{3x-2} = (3x-2)^{\frac{1}{2}}$ **c** $(a+b)^{\frac{1}{7}} = \sqrt[7]{a+b}$

Further fractional indices

$$a^{-\frac{1}{n}} = \frac{1}{\sqrt[n]{a}}$$

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} \text{ or } \left(\sqrt[n]{a}\right)^m$$

Proof

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = \left(\sqrt[n]{a}\right)^m$$

$$a^{\frac{m}{n}} = \left(a^m\right)^{\frac{1}{n}} = \sqrt[n]{a^m}$$

ISBN 9780170413152

$$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^{n}$$

Proof

$$\begin{aligned}
\left(\frac{a}{b}\right)^{-n} &= \frac{1}{\left(\frac{a}{b}\right)^{n}} \\
&= \frac{1}{\frac{a^n}{b^n}} \\
&= 1 \div \frac{a^n}{b^n} \\
&= 1 \times \frac{b^n}{a^n} \\
&= \frac{b^n}{a^n} \\
&= \left(\frac{b}{a}\right)^{n}
\end{aligned}$$

EXAMPLE 4

a Evaluate:

i $8^{\frac{4}{3}}$ **ii** $125^{-\frac{1}{3}}$ **iii** $\left(\frac{2}{3}\right)^{-3}$

b Write in index form:

i $\sqrt{x^5}$ **ii** $\dfrac{1}{\sqrt[3]{\left(4x^2-1\right)^2}}$

c Write $r^{-\frac{3}{5}}$ without the negative and fractional indices.

Solution

a **i** $8^{\frac{4}{3}} = \left(\sqrt[3]{8}\right)^4 \left(\text{or } \sqrt[3]{8^4}\right)$
$= 2^4$
$= 16$

ii $125^{-\frac{1}{3}} = \dfrac{1}{125^{\frac{1}{3}}}$
$= \dfrac{1}{\sqrt[3]{125}}$
$= \dfrac{1}{5}$

iii $\left(\dfrac{2}{3}\right)^{-3} = \left(\dfrac{3}{2}\right)^3$
$= \dfrac{27}{8}$
$= 3\frac{3}{8}$

b **i** $\sqrt{x^5} = x^{\frac{5}{2}}$

ii $\dfrac{1}{\sqrt[3]{(4x^2-1)^2}} = \dfrac{1}{(4x^2-1)^{\frac{2}{3}}}$
$= (4x^2-1)^{-\frac{2}{3}}$

c $r^{-\frac{3}{5}} = \dfrac{1}{r^{\frac{3}{5}}} = \dfrac{1}{\sqrt[5]{r^3}}$

DID YOU KNOW?

Fractional indices

Nicole Oresme (1323–82) was the first mathematician to use fractional indices.

John Wallis (1616–1703) was the first person to explain the significance of zero, negative and fractional indices. He also introduced the symbol ∞ for infinity.

Research these mathematicians and find out more about their work and backgrounds. You could use keywords such as indices and infinity as well as their names to find this information.

Exercise 1.03 Fractional indices

1 Evaluate:

a $81^{\frac{1}{2}}$ **b** $27^{\frac{1}{3}}$ **c** $16^{\frac{1}{2}}$ **d** $8^{\frac{1}{3}}$ **e** $49^{\frac{1}{2}}$

f $1000^{\frac{1}{3}}$ **g** $16^{\frac{1}{4}}$ **h** $64^{\frac{1}{2}}$ **i** $64^{\frac{1}{3}}$ **j** $1^{\frac{1}{7}}$

k $81^{\frac{1}{4}}$ **l** $32^{\frac{1}{5}}$ **m** $0^{\frac{1}{8}}$ **n** $125^{\frac{1}{3}}$ **o** $343^{\frac{1}{3}}$

p $128^{\frac{1}{7}}$ **q** $256^{\frac{1}{4}}$ **r** $125^{\frac{2}{3}}$ **s** $4^{\frac{5}{2}}$ **t** $8^{\frac{2}{3}}$

u $9^{\frac{3}{2}}$ **v** $8^{-\frac{1}{3}}$ **w** $9^{-\frac{1}{2}}$ **x** $16^{-\frac{1}{4}}$ **y** $64^{-\frac{2}{3}}$

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2 Evaluate correct to 2 decimal places:

a $23^{\frac{1}{4}}$ **b** $\sqrt[4]{45.8}$ **c** $\sqrt[7]{1.24+4.3^2}$

d $\dfrac{1}{\sqrt[5]{12.9}}$ **e** $\sqrt[8]{\dfrac{3.6-1.4}{1.5+3.7}}$ **f** $\dfrac{\sqrt[4]{5.9\times 3.7}}{8.79-1.4}$

3 Write without fractional or negative indices:

a $y^{\frac{1}{3}}$ **b** $x^{\frac{1}{6}}$ **c** $a^{\frac{1}{2}}$ **d** $t^{\frac{1}{9}}$ **e** $y^{\frac{2}{3}}$

f $x^{\frac{3}{4}}$ **g** $b^{\frac{2}{5}}$ **h** $a^{\frac{4}{7}}$ **i** $x^{-\frac{1}{2}}$ **j** $d^{-\frac{1}{3}}$

k $x^{-\frac{1}{8}}$ **l** $y^{-\frac{1}{3}}$ **m** $a^{-\frac{1}{4}}$ **n** $z^{-\frac{3}{4}}$ **o** $y^{-\frac{3}{5}}$

p $(2x+5)^{\frac{1}{2}}$ **q** $(6q+r)^{\frac{1}{3}}$ **r** $(a+b)^{\frac{1}{9}}$ **s** $(3x-1)^{-\frac{1}{2}}$ **t** $(x+7)^{-\frac{2}{5}}$

4 Write in index form:

a $\sqrt{t}$ **b** $\sqrt[5]{y}$ **c** $\sqrt{x^3}$ **d** $\sqrt[3]{9-x}$ **e** $\sqrt{4s+1}$

f $\sqrt{(3x+1)^5}$ **g** $\dfrac{1}{\sqrt{2t+3}}$ **h** $\dfrac{1}{\sqrt{(5x-y)^3}}$ **i** $\dfrac{1}{\sqrt[3]{(x-2)^2}}$ **j** $\dfrac{1}{2\sqrt{y+7}}$

k $\dfrac{5}{\sqrt[3]{x+4}}$ **l** $\dfrac{1}{3\sqrt{y^2-1}}$ **m** $\dfrac{3}{5\sqrt[4]{(x^2+2)^3}}$

5 Write in index form and simplify:

a $x\sqrt{x}$ **b** $\dfrac{\sqrt{x}}{x}$ **c** $\dfrac{x}{\sqrt[3]{x}}$ **d** $\dfrac{x^2}{\sqrt[3]{x}}$ **e** $x\sqrt[4]{x}$

6 Write without fractional or negative indices:

a $(a-2b)^{-\frac{1}{3}}$ **b** $(y-3)^{-\frac{2}{3}}$ **c** $4(6a+1)^{-\frac{4}{7}}$ **d** $\dfrac{(x+y)^{-\frac{5}{4}}}{3}$ **e** $\dfrac{6(3x+8)^{-\frac{2}{9}}}{7}$

DID YOU KNOW?

The beginnings of algebra

One of the earliest mathematicians to use algebra was **Diophantus of Alexandria** in Greece. It is not known when he lived, but it is thought this may have been around 250 CE.

In Persia around 700–800 CE a mathematician named **Muhammad ibn Musa al-Khwarizmi** wrote books on algebra and Hindu numerals. One of his books was named *Al-Jabr wa'l Muqabala*, and the word **algebra** comes from the first word in this title.

1.04 Simplifying algebraic expressions

EXAMPLE 5

Simplify:

a $4x^2 - 3x^2 + 6x^2$ **b** $x^3 - 3x - 5x + 4$ **c** $3a - 4b - 5a - b$

Solution

a $4x^2 - 3x^2 + 6x^2 = x^2 + 6x^2$
$= 7x^2$

b $x^3 - 3x - 5x + 4 = x^3 - 8x + 4$

Only 'like' terms can be added or subtracted.

c $3a - 4b - 5a - b = 3a - 5a - 4b - b$
$= -2a - 5b$

EXAMPLE 6

Simplify:

a $-5x \times 3y \times 2x$

b $\dfrac{5a^3b}{15ab^2}$

Solution

a $-5x \times 3y \times 2x = -30xyx$
$= -30x^2y$

b $\dfrac{5a^3b}{15ab^2} = \dfrac{1}{3}a^{3-1}b^{1-2}$
$= \dfrac{1}{3}a^2b^{-1}$
$= \dfrac{a^2}{3b}$

ISBN 9780170413152

Exercise 1.04 Simplifying algebraic expressions

1 Simplify:

a $9a - 6a$ **b** $5z - 4z$ **c** $4b - b$

d $2r - 5r$ **e** $-4y + 3y$ **f** $-2x - 3x$

g $2a - 2a$ **h** $-4k + 7k$ **i** $3t + 4t + 2t$

j $8w - w + 3w$ **k** $4m - 3m - 2m$ **l** $x + 3x - 5x$

m $8h - h - 7h$ **n** $3b - 5b + 4b + 9b$ **o** $-5x + 3x - x - 7x$

p $6x - 5y - y$ **q** $8a + b - 4b - 7a$ **r** $xy + 2y + 3xy$

s $2ab^2 - 5ab^2 - 3ab^2$ **t** $m^2 - 5m - m + 12$ **u** $p^2 - 7p + 5p - 6$

v $ab + 2b - 3ab + 8b$ **w** $ab + bc - ab - ac + bc$

x $a^5 - 7x^3 + a^5 - 2x^3 + 1$ **y** $x^3 - 3xy^2 + 4x^2y - x^2y + xy^2 + 2y^3$

2 Simplify:

a $5 \times 2b$ **b** $2x \times 4y$ **c** $5p \times 2p$

d $-3z \times 2w$ **e** $-5a \times -3b$ **f** $x \times 2y \times 7z$

g $8ab \times 6c$ **h** $4d \times 3d$ **i** $3a \times 4a \times a$

j $(-3y)^3$ **k** $(2x^2)^5$ **l** $2ab^3 \times 3a$

m $5a^2b \times -2ab$ **n** $7pq^2 \times 3p^2q^2$ **o** $5ab \times a^2b^2$

p $4h^3 \times -2h^7$ **q** $k^3p \times p^2$ **r** $(-3t^3)^4$

s $7m^6 \times -2m^5$ **t** $-2x^2 \times 3x^3y \times -4xy^2$

3 Simplify:

a $30x \div 5$ **b** $2y \div y$ **c** $\dfrac{8a^2}{2}$

d $\dfrac{8a^2}{a}$ **e** $\dfrac{8a^2}{2a}$ **f** $\dfrac{xy}{2x}$

g $12p^3 \div 4p^2$ **h** $\dfrac{3a^2b^2}{6ab}$ **i** $\dfrac{20x}{15xy}$

j $\dfrac{-9x^7}{3x^4}$ **k** $-15ab \div -5b$ **l** $\dfrac{2ab}{6a^2b^3}$

m $\dfrac{-8p}{4pqs}$ **n** $14cd^2 \div 21c^3d^3$ **o** $\dfrac{2xy^2z^3}{4x^3y^2z}$

p $\dfrac{42p^5q^4}{7pq^3}$ **q** $5a^9b^4c^{-2} \div 20a^5b^{-3}c^{-1}$ **r** $\dfrac{2(a^{-5})^2b^4}{4a^{-9}(b^2)^{-1}}$

s $-5x^4y^7z \div 15xy^8z^{-2}$ **t** $-9(a^4b^{-1})^3 \div -18a^{-1}b^3$

1.05 Expansion

WS
Expanding algebraic expressions

When we remove grouping symbols we say that we are **expanding** an expression.

Expanding expressions

To expand an expression, use the distributive law:

$$a(b + c) = ab + ac$$

EXAMPLE 7

Expand and simplify:

a $5a^2(4 + 3ab - c)$ **b** $5 - 2(y + 3)$ **c** $2(b - 5) - (b + 1)$

Solution

a
$$\begin{aligned} 5a^2(4 + 3ab - c) &= 5a^2 \times 4 + 5a^2 \times 3ab - 5a^2 \times c \\ &= 20a^2 + 15a^3b - 5a^2c \end{aligned}$$

b
$$\begin{aligned} 5 - 2(y + 3) &= 5 - 2 \times y - 2 \times 3 \\ &= 5 - 2y - 6 \\ &= -2y - 1 \end{aligned}$$

c
$$\begin{aligned} 2(b - 5) - (b + 1) &= 2 \times b + 2 \times -5 - 1 \times b - 1 \times 1 \\ &= 2b - 10 - b - 1 \\ &= b - 11 \end{aligned}$$

Exercise 1.05 Expansion

Expand and simplify each expression.

1 $2(x - 4)$ **2** $3(2h + 3)$ **3** $-5(a - 2)$

4 $x(2y + 3)$ **5** $x(x - 2)$ **6** $2a(3a - 8b)$

7 $ab(2a + b)$ **8** $5n(n - 4)$ **9** $3x^2y(xy + 2y^2)$

10 $3 + 4(k + 1)$ **11** $2(t - 7) - 3$ **12** $y(4y + 3) + 8y$

13 $9 - 5(b + 3)$ **14** $3 - (2x - 5)$ **15** $5(3 - 2m) + 7(m - 2)$

16 $2(h + 4) + 3(2h - 9)$ **17** $3(2d - 3) - (5d - 3)$ **18** $a(2a + 1) - (a^2 + 3a - 4)$

19 $x(3x - 4) - 5(x + 1)$ **20** $2ab(3 - a) - b(4a - 1)$ **21** $5x - (x - 2) - 3$

22 $8 - 4(2y + 1) + y$ **23** $(a + b) - (a - b)$ **24** $2(3t - 4) - (t + 1) + 3$

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1.06 Binomial products

Expanding binomial products

A **binomial expression** consists of 2 **terms**; for example, $x + 3$.

A set of 2 binomial expressions multiplied together is called a **binomial product**; for example, $(x + 3)(x - 2)$.

Each term in the first bracket is multiplied by each term in the second bracket.

Binomial product

$$(x + a)(x + b) = x^2 + bx + ax + ab$$

EXAMPLE 8

Expand and simplify:

a $(p + 3)(q - 4)$ **b** $(a + 5)^2$ **c** $(x + 4)(2x - 3y - 1)$

Solution

a $(p + 3)(q - 4) = pq - 4p + 3q - 12$

b $(a + 5)^2 = (a + 5)(a + 5)$
$= a^2 + 5a + 5a + 25$
$= a^2 + 10a + 25$

c $(x + 4)(2x - 3y - 1) = 2x^2 - 3xy - x + 8x - 12y - 4$
$= 2x^2 - 3xy + 7x - 12y - 4$

Exercise 1.06 Binomial products

Expand and simplify:

1 $(a + 5)(a + 2)$ **2** $(x + 3)(x - 1)$ **3** $(2y - 3)(y + 5)$ **4** $(m - 4)(m - 2)$

5 $(x + 4)(x + 3)$ **6** $(y + 2)(y - 5)$ **7** $(2x - 3)(x + 2)$ **8** $(h - 7)(h - 3)$

9 $(x + 5)(x - 5)$ **10** $(5a - 4)(3a - 1)$ **11** $(2y + 3)(4y - 3)$ **12** $(x - 4)(y + 7)$

13 $(x^2 + 3)(x - 2)$ **14** $(n + 2)(n - 2)$ **15** $(2x + 3)(2x - 3)$ **16** $(4 - 7y)(4 + 7y)$

17 $(a + 2b)(a - 2b)$ **18** $(3x - 4y)(3x + 4y)$ **19** $(x + 3)(x - 3)$ **20** $(y - 6)(y + 6)$

21 $(3a + 1)(3a - 1)$ **22** $(2z - 7)(2z + 7)$ **23** $(x + 9)(x - 2y + 2)$

24 $(b - 3)(2a + 2b - 1)$ **25** $(x + 2)(x^2 - 2x + 4)$ **26** $(a - 3)(a^2 + 3a + 9)$

27 $(a + 9)^2$ **28** $(k - 4)^2$ **29** $(x + 2)^2$ **30** $(y - 7)^2$

31 $(2x + 3)^2$ **32** $(2t - 1)^2$ **33** $(3a + 4b)^2$ **34** $(x - 5y)^2$

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35 $(2a+b)^2$ **36** $(a-b)(a+b)$ **37** $(a+b)^2$ **38** $(a-b)^2$

39 $(a+b)(a^2-ab+b^2)$ **40** $(a-b)(a^2+ab+b^2)$

WS Expanding expressions

Special binomial products

1.07 Special products

Some binomial products have special results and can be simplified quickly using their special properties. Did you notice some of these in Exercise 1.06?

Difference of two squares

$$(a+b)(a-b)=a^2-b^2$$

Perfect squares

$$(a+b)^2=a^2+2ab+b^2$$

$$(a-b)^2=a^2-2ab+b^2$$

EXAMPLE 9

Expand and simplify:

a $(2x-3)^2$ **b** $(3y-4)(3y+4)$

Solution

a $(2x-3)^2=(2x)^2-2(2x)3+3^2$
$\qquad=4x^2-12x+9$

b $(3y-4)(3y+4)=(3y)^2-4^2$
$\qquad=9y^2-16$

Exercise 1.07 Special products

Expand and simplify:

1 $(t+4)^2$ **2** $(z-6)^2$ **3** $(x-1)^2$

4 $(y+8)^2$ **5** $(q+3)^2$ **6** $(k-7)^2$

7 $(n+1)^2$ **8** $(2b+5)^2$ **9** $(3-x)^2$

10 $(3y-1)^2$ **11** $(x+y)^2$ **12** $(3a-b)^2$

13 $(4d+5e)^2$ **14** $(t+4)(t-4)$ **15** $(x-3)(x+3)$

16 $(p+1)(p-1)$ **17** $(r+6)(r-6)$ **18** $(x-10)(x+10)$

19 $(2a+3)(2a-3)$ **20** $(x-5y)(x+5y)$ **21** $(4a+1)(4a-1)$

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22 $(7-3x)(7+3x)$ **23** $(x^2+2)(x^2-2)$ **24** $(x^2+5)^2$

25 $(3ab-4c)(3ab+4c)$ **26** $\left(x+\frac{2}{x}\right)^2$ **27** $\left(a-\frac{1}{a}\right)\left(a+\frac{1}{a}\right)$

28 $[x+(y-2)][x-(y-2)]$ **29** $[(a+b)+c]^2$ **30** $[(x+1)-y]^2$

31 $(a+3)^2-(a-3)^2$ **32** $16-(z-4)(z+4)$ **33** $2x+(3x+1)^2-4$

34 $(x+y)^2-x(2-y)$ **35** $(4n-3)(4n+3)-2n^2+5$ **36** $(x-4)^3$

37 $\left(x-\frac{1}{x}\right)^2-\left(\frac{1}{x}\right)^2+2$ **38** $(x^2+y^2)^2-4x^2y^2$ **39** $(2a+5)^3$

1.08 Factorisation

WS Factorising algebraic expressions

Factors divide exactly into an equal or larger number or term, without leaving a remainder.

Factorising

To **factorise** an expression, we use the distributive law in the opposite way from when we expand brackets.

$$ax+bx=x(a+b)$$

EXAMPLE 10

Factorise:

a $3x+12$ **b** y^2-2y **c** x^3-2x^2

d $5(x+3)+2y(x+3)$ **e** $8a^3b^2-2ab^3$

Solution

a The highest common factor is 3. $3x+12=3(x+4)$

b The highest common factor is y. $y^2-2y=y(y-2)$

c x and x^2 are both common factors. Take out the highest common factor, which is x^2. $x^3-2x^2=x^2(x-2)$

d The highest common factor is $x+3$. $5(x+3)+2y(x+3)=(x+3)(5+2y)$

e The highest common factor is $2ab^2$. $8a^3b^2-2ab^3=2ab^2(4a^2-b)$

ISBN 9780170413152

Exercise 1.08 Factorisation

Factorise:

1 $2y + 6$

2 $5x - 10$

3 $3m - 9$

4 $8x + 2$

5 $24 - 18y$

6 $x^2 + 2x$

7 $m^2 - 3m$

8 $2y^2 + 4y$

9 $15a - 3a^2$

10 $ab^2 + ab$

11 $4x^2y - 2xy$

12 $3mn^3 + 9mn$

13 $8x^2z - 2xz^2$

14 $6ab + 3a - 2a^2$

15 $5x^2 - 2x + xy$

16 $3q^5 - 2q^2$

17 $5b^3 + 15b^2$

18 $6a^2b^3 - 3a^3b^2$

19 $x(m + 5) + 7(m + 5)$

20 $2(y - 1) - y(y - 1)$

21 $4(7 + y) - 3x(7 + y)$

22 $6x(a - 2) + 5(a - 2)$

23 $x(2t + 1) - y(2t + 1)$

24 $a(3x - 2) + 2b(3x - 2) - 3c(3x - 2)$

25 $6x^3 + 9x^2$

26 $3pq^5 - 6q^3$

27 $15a^4b^3 + 3ab$

28 $4x^3 - 24x^2$

29 $35m^3n^4 - 25m^2n$

30 $24a^2b^5 + 16ab^2$

31 $2\pi r^2 + 2\pi rh$

32 $(x - 3)^2 + 5(x - 3)$

33 $y^2(x + 4) + 2(x + 4)$

34 $a(a + 1) - (a + 1)^2$

1.09 Factorisation by grouping in pairs

Factorising by grouping in pairs

If an expression has 4 terms, it can sometimes be factorised in pairs.

$$\begin{aligned} ax + bx + ay + by &= x(a + b) + y(a + b) \\ &= (a + b)(x + y) \end{aligned}$$

EXAMPLE 11

Factorise:

a $x^2 - 2x + 3x - 6$

b $2x - 4 + 6y - 3xy$

Solution

a
$$\begin{aligned} x^2 - 2x + 3x - 6 &= x(x - 2) + 3(x - 2) \\ &= (x - 2)(x + 3) \end{aligned}$$

b
$$\begin{aligned} 2x - 4 + 6y - 3xy &= 2(x - 2) + 3y(2 - x) \\ &= 2(x - 2) - 3y(x - 2) \\ &= (x - 2)(2 - 3y) \end{aligned}$$

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Exercise 1.09 Factorisation by grouping in pairs

Factorise:

1 $2x + 8 + bx + 4b$

2 $ay - 3a + by - 3b$

3 $x^2 + 5x + 2x + 10$

4 $m^2 - 2m + 3m - 6$

5 $ad - ac + bd - bc$

6 $x^3 + x^2 + 3x + 3$

7 $5ab - 3b + 10a - 6$

8 $2xy - x^2 + 2y^2 - xy$

9 $ay + a + y + 1$

10 $x^2 + 5x - x - 5$

11 $y + 3 + ay + 3a$

12 $m - 2 + 4y - 2my$

13 $2x^2 + 10xy - 3xy - 15y^2$

14 $a^2b + ab^3 - 4a - 4b^2$

15 $5x - x^2 - 3x + 15$

16 $x^4 + 7x^3 - 4x - 28$

17 $7x - 21 - xy + 3y$

18 $4d + 12 - de - 3e$

19 $3x - 12 + xy - 4y$

20 $2a + 6 - ab - 3b$

21 $x^3 - 3x^2 + 6x - 18$

22 $pq - 3p + q^2 - 3q$

23 $3x^3 - 6x^2 - 5x + 10$

24 $4a - 12b + ac - 3bc$

25 $xy + 7x - 4y - 28$

26 $x^4 - 4x^3 - 5x + 20$

27 $4x^3 - 6x^2 + 8x - 12$

28 $3a^2 + 9a + 6ab + 18b$

29 $5y - 15 + 10xy - 30x$

30 $\pi r^2 + 2\pi r - 3r - 6$

1.10 Factorising trinomials

Factorising quadratic expressions

A **trinomial** is an expression with 3 terms; for example, $x^2 - 4x + 3$. Factorising a trinomial usually gives a **binomial product**.

We know that: $(x + a)(x + b) = x^2 + bx + ax + ab$

$$= x^2 + (a + b)x + ab$$

Factorising trinomials

$$x^2 + (a + b)x + ab = (x + a)(x + b)$$

Find values for a and b so that the sum $a + b$ is the middle term and the product ab is the last term.

EXAMPLE 12

Factorise:

a $m^2 - 5m + 6$

b $y^2 + y - 2$

Solution

a $a + b = -5$ and $ab = 6$

To have $a + b = -5$, at least one number must be negative.

To have $ab = 6$, both numbers have the same sign. So both are negative.

For $ab = 6$: we could have $-6 \times (-1)$ or $-3 \times (-2)$

$-3 + (-2) = -5$ so $a = -3$ and $b = -2$.

So $m^2 - 5m + 6 = (m - 3)(m - 2)$

Check: $(m - 3)(m - 2) = m^2 - 2m - 3m + 6$

$= m^2 - 5m + 6$

b $a + b = 1$ and $ab = -2$

To have $ab = -2$, the numbers must have opposite signs. So one is positive and one is negative.

For $ab = -2$: we could have -2×1 or -1×2

$-1 + 2 = 1$ so $a = -1$ and $b = 2$.

So $y^2 + y - 2 = (y - 1)(y + 2)$

Check: $(y - 1)(y + 2) = y^2 + 2y - y - 2$

$= y^2 + y - 2$

Exercise 1.10 Factorising trinomials

Factorise:

1 $x^2 + 4x + 3$

2 $y^2 + 7y + 12$

3 $m^2 + 2m + 1$

4 $t^2 + 8t + 16$

5 $z^2 + z - 6$

6 $x^2 - 5x - 6$

7 $v^2 - 8v + 15$

8 $t^2 - 6t + 9$

9 $x^2 + 9x - 10$

10 $y^2 - 10y + 21$

11 $m^2 - 9m + 18$

12 $y^2 + 9y - 36$

13 $x^2 - 5x - 24$

14 $a^2 - 4a + 4$

15 $x^2 + 14x - 32$

16 $y^2 - 5y - 36$

17 $n^2 - 10n + 24$

18 $x^2 - 10x + 25$

19 $p^2 + 8p - 9$

20 $k^2 - 7k + 10$

21 $x^2 + x - 12$

22 $m^2 - 6m - 7$

23 $q^2 + 12q + 20$

24 $d^2 - 4d - 5$

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1.11 Further trinomials

When the coefficient of the first term is not 1, for example $5x^2 - 13x + 6$, we need to use a different method to factorise the trinomial.

The coefficient of the first term is the number in front of the x^2.

This method still involves finding 2 numbers that give a required sum and product but it also involves grouping in pairs.

Factorising quadratic expressions (Advanced)

WS

Excel worksheet: Factorising trinomials

Excel spreadsheet: Factorising trinomials

EXAMPLE 13

Factorise:

a $5x^2 - 13x + 6$

b $4y^2 + 4y - 3$

Solution

a First, multiply the coefficient of the first term by the last term: $5 \times 6 = 30$.

Now $a + b = -13$ and $ab = 30$.

Since the sum is negative and the product is positive, a and b must be both negative.

2 numbers with product 30 and sum -13 are -10 and -3.

Now write the trinomial with the middle term split into 2 terms $-10x$ and $-3x$, and then factorise by grouping in pairs.

$$\begin{aligned} 5x^2 - 13x + 6 &= 5x^2 - 10x - 3x + 6 \\ &= 5x(x - 2) - 3(x - 2) \end{aligned}$$

If you factorise correctly, you should always find a common factor remaining, such as $(x - 2)$ here.

$$= (x - 2)(5x - 3)$$

b First, multiply the coefficient of the first term by the last term: $4(-3) = -12$

Now $a + b = 4$ and $ab = -12$.

Since the product is negative, a and b have opposite signs (one positive and one negative).

2 numbers with product -12 and sum 4 are 6 and -2.

Now write the trinomial with the middle term split into 2 terms $6y$ and $-2y$, and then factorise by grouping in pairs.

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$$4y^2 + 4y - 3 = 4y^2 + 6y - 2y - 3$$
$$= 2y(2y + 3) - 1(2y + 3)$$
$$= (2y + 3)(2y - 1)$$

There are other ways of factorising these trinomials. Your teacher may show you some of these.

Exercise 1.11 Further trinomials

Factorise:

1 $2a^2 + 11a + 5$

2 $5y^2 + 7y + 2$

3 $3x^2 + 10x + 7$

4 $3x^2 + 8x + 4$

5 $2b^2 - 5b + 3$

6 $7x^2 - 9x + 2$

7 $3y^2 + 5y - 2$

8 $2x^2 + 11x + 12$

9 $5p^2 + 13p - 6$

10 $6x^2 + 13x + 5$

11 $2y^2 - 11y - 6$

12 $10x^2 + 3x - 1$

13 $8t^2 - 14t + 3$

14 $6x^2 - x - 12$

15 $6y^2 + 47y - 8$

16 $4n^2 - 11n + 6$

17 $8t^2 + 18t - 5$

18 $12q^2 + 23q + 10$

19 $4r^2 + 11r - 3$

20 $4x^2 - 4x - 15$

21 $6y^2 - 13y + 2$

22 $6p^2 - 5p - 6$

23 $8x^2 + 31x + 21$

24 $12b^2 - 43b + 36$

25 $6x^2 - 53x - 9$

26 $9x^2 + 30x + 25$

27 $16y^2 + 24y + 9$

28 $25k^2 - 20k + 4$

29 $36a^2 - 12a + 1$

30 $49m^2 + 84m + 36$

1.12 Perfect squares

You have looked at expanding $(a + b)^2 = a^2 + 2ab + b^2$ and $(a - b)^2 = a^2 - 2ab + b^2$. These are called **perfect squares**.

When factorising, use these results the other way around.

EXAMPLE 14

Factorise:

a $x^2 - 8x + 16$

b $4a^2 + 20a + 25$

Solution

a
$$x^2 - 8x + 16 = x^2 - 2(4)x + 4^2$$
$$= (x - 4)^2$$

b
$$4a^2 + 20a + 25 = (2a)^2 + 2(2a)(5) + 5^2$$
$$= (2a + 5)^2$$

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Exercise 1.12 Perfect squares

Factorise:

1 $y^2 - 2y + 1$

2 $x^2 + 6x + 9$

3 $m^2 + 10m + 25$

4 $t^2 - 4t + 4$

5 $x^2 - 12x + 36$

6 $4x^2 + 12x + 9$

7 $16b^2 - 8b + 1$

8 $9a^2 + 12a + 4$

9 $25x^2 - 40x + 16$

10 $49y^2 + 14y + 1$

11 $9y^2 - 30y + 25$

12 $16k^2 - 24k + 9$

13 $25x^2 + 10x + 1$

14 $81a^2 - 36a + 4$

15 $49m^2 + 84m + 36$

16 $t^2 + t + \frac{1}{4}$

17 $x^2 - \frac{4x}{3} + \frac{4}{9}$

18 $9y^2 + \frac{6y}{5} + \frac{1}{25}$

19 $x^2 + 2 + \frac{1}{x^2}$

20 $25k^2 - 20 + \frac{4}{k^2}$

1.13 Difference of two squares

Difference of two squares

$$a^2 - b^2 = (a + b)(a - b)$$

EXAMPLE 15

Factorise:

a $d^2 - 36$

b $1 - 9b^2$

c $(a + 3)^2 - (b - 1)^2$

Solution

a $d^2 - 36 = d^2 - 6^2$

$= (d + 6)(d - 6)$

b $1 - 9b^2 = 1^2 - (3b)^2$

$= (1 + 3b)(1 - 3b)$

c $(a + 3)^2 - (b - 1)^2 = [(a + 3) + (b - 1)][(a + 3) - (b - 1)]$

$= (a + 3 + b - 1)(a + 3 - b + 1)$

$= (a + b + 2)(a - b + 4)$

Exercise 1.13 Difference of two squares

Factorise:

1 $a^2 - 4$

2 $x^2 - 9$

3 $y^2 - 1$

4 $x^2 - 25$

5 $4x^2 - 49$

6 $16y^2 - 9$

7 $1 - 4z^2$

8 $25t^2 - 1$

9 $9t^2 - 4$

10 $9 - 16x^2$

11 $x^2 - 4y^2$

12 $36x^2 - y^2$

13 $4a^2 - 9b^2$

14 $x^2 - 100y^2$

15 $4a^2 - 81b^2$

16 $(x + 2)^2 - y^2$

17 $(a - 1)^2 - (b - 2)^2$

18 $z^2 - (1 + w)^2$

19 $x^2 - \frac{1}{4}$

20 $\frac{y^2}{9} - 1$

21 $(x + 2)^2 - (2y + 1)^2$

22 $x^4 - 1$

23 $9x^6 - 4y^2$

24 $x^4 - 16y^4$

1.14 Mixed factorisation

WS

Factorising expressions

EXAMPLE 16

Factorise $5x^2 - 45$.

Solution

Using simple factors: $5x^2 - 45 = 5(x^2 - 9)$

The difference of 2 squares: $= 5(x + 3)(x - 3)$

Exercise 1.14 Mixed factorisation

Factorise:

1 $4a^3 - 36a$

2 $2x^2 - 18$

3 $3p^2 - 3p - 36$

4 $5y^2 - 5$

5 $5a^2 - 10a + 5$

6 $3z^3 + 27z^2 + 60z$

7 $9ab - 4a^3b^3$

8 $x^3 - x$

9 $6x^2 + 8x - 8$

10 $y^2(y + 5) - 16(y + 5)$

11 $x^4 + 8x^3 - x^2 - 8x$

12 $y^6 - 4$

13 $x^3 - 3x^2 - 10x$

14 $x^3 - 3x^2 - 9x + 27$

15 $4x^2y^3 - y$

16 $24 - 6b^2$

17 $18x^2 + 33x - 30$

18 $3x^2 - 6x + 3$

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19 $x^3 + 2x^2 - 25x - 50$

20 $z^3 + 6z^2 + 9z$

21 $3y^2 + 30y + 75$

22 $ab^2 - 9a$

23 $4k^3 + 40k^2 + 100k$

24 $3x^3 + 9x^2 - 3x - 9$

25 $4a^3b + 8a^2b^2 - 4ab^2 - 2a^2b$

1.15 Simplifying algebraic fractions

EXAMPLE 17

Simplify:

a $\dfrac{4x+2}{2}$

b $\dfrac{2x^2-3x-2}{x^2-4}$

Solution

a $\dfrac{4x+2}{2} = \dfrac{2(2x+1)}{2}$

$= 2x + 1$

b Factorise both top and bottom.

$\dfrac{2x^2-3x-2}{x^2-4} = \dfrac{(2x+1)(x-2)}{(x-2)(x+2)}$

$= \dfrac{2x+1}{x+2}$

Exercise 1.15 Simplifying algebraic fractions

Simplify:

1 $\dfrac{5a+10}{5}$

2 $\dfrac{6t-3}{3}$

3 $\dfrac{8y+2}{6}$

4 $\dfrac{8}{4d-2}$

5 $\dfrac{x^2}{5x^2-2x}$

6 $\dfrac{y-4}{y^2-8y+16}$

7 $\dfrac{2ab-4a^2}{a^2-3a}$

8 $\dfrac{s^2+s-2}{s^2+5s+6}$

9 $\dfrac{b^4-1}{b^2-1}$

10 $\dfrac{2p^2+7p-15}{6p-9}$

11 $\dfrac{a^2-1}{a^2+2a-3}$

12 $\dfrac{3(x-2)+y(x-2)}{x^2-4}$

13 $\dfrac{x^3+3x^2-9x-27}{x^2+6x+9}$

14 $\dfrac{2p^2-3p-2}{2p^2+p}$

15 $\dfrac{ay-ax+by-bx}{2ay-by-2ax+bx}$

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WS Algebraic fractions

1.16 Operations with algebraic fractions

EXAMPLE 18

Simplify:

a $\dfrac{x-1}{5}-\dfrac{x+3}{4}$ **b** $\dfrac{2a^2b+10ab}{b^2-9}\div\dfrac{a^2-25}{4b+12}$ **c** $\dfrac{2}{x-5}+\dfrac{1}{x+2}$ **d** $\dfrac{2}{x+1}-\dfrac{1}{x^2-1}$

Solution

a
$$\begin{aligned}\frac{x-1}{5}-\frac{x+3}{4}&=\frac{4(x-1)-5(x+3)}{20}\\&=\frac{4x-4-5x-15}{20}\\&=\frac{-x-19}{20}\end{aligned}$$

b
$$\begin{aligned}\frac{2a^2b+10ab}{b^2-9}\div\frac{a^2-25}{4b+12}&=\frac{2a^2b+10ab}{b^2-9}\times\frac{4b+12}{a^2-25}\\&=\frac{2ab(a+5)}{(b+3)(b-3)}\times\frac{4(b+3)}{(a+5)(a-5)}\\&=\frac{8ab}{(a-5)(b-3)}\end{aligned}$$

c
$$\begin{aligned}\frac{2}{x-5}+\frac{1}{x+2}&=\frac{2(x+2)+1(x-5)}{(x-5)(x+2)}\\&=\frac{2x+4+x-5}{(x-5)(x+2)}\\&=\frac{3x-1}{(x-5)(x+2)}\end{aligned}$$

d
$$\begin{aligned}\frac{2}{x+1}-\frac{1}{x^2-1}&=\frac{2}{x+1}-\frac{1}{(x+1)(x-1)}\\&=\frac{2(x-1)}{(x+1)(x-1)}-\frac{1}{(x+1)(x-1)}\\&=\frac{2x-2}{(x+1)(x-1)}-\frac{1}{(x+1)(x-1)}\\&=\frac{2x-2-1}{(x+1)(x-1)}\\&=\frac{2x-3}{(x+1)(x-1)}\end{aligned}$$

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Exercise 1.16 Operations with algebraic fractions

1 Simplify:

a $\frac{x}{2}+\frac{3x}{4}$ **b** $\frac{y+1}{5}+\frac{2y}{3}$ **c** $\frac{a+2}{3}-\frac{a}{4}$

d $\frac{p-3}{6}+\frac{p+2}{2}$ **e** $\frac{x-5}{2}-\frac{x-1}{3}$

2 Simplify:

a $\frac{3x+6}{5}\times\frac{10}{x+2}$ **b** $\frac{a^2-4}{3}\times\frac{5b}{a+2}$

c $\frac{t^2+3t-10}{xy^2}\div\frac{5t-10}{2xy}$ **d** $\frac{2a-6}{2x+4}\times\frac{5x+10}{4}$

e $\frac{5x+10-xy-2y}{15}\div\frac{7x+14}{3}$ **f** $\frac{3}{b+2}\times\frac{b^2+2b}{6a-3}$

g $\frac{3ab^2}{5xy}\div\frac{12ab-6a}{x^2y+2xy^2}$ **h** $\frac{ax-ay+bx-by}{x^2-y^2}\times\frac{x^2y+xy^2}{ab^2+a^2b}$

i $\frac{x^2-6x+9}{x^2-25}\div\frac{x^2-5x+6}{x^2+4x-5}$ **j** $\frac{p^2-4}{q^2+2q+1}\times\frac{5q+5}{3p+6}$

3 Simplify:

a $\frac{2}{x}+\frac{3}{x}$ **b** $\frac{1}{x-1}-\frac{2}{x}$ **c** $1+\frac{3}{a+b}$

d $x-\frac{x^2}{x+2}$ **e** $p-q+\frac{1}{p+q}$ **f** $\frac{1}{x+1}+\frac{1}{x-3}$

g $\frac{2}{x^2-4}-\frac{3}{x+2}$ **h** $\frac{1}{a^2+2a+1}+\frac{1}{a+1}$

4 Simplify:

a $\frac{a^2-5a}{y^2-4y+4}\div\frac{3a-15}{y^2-4}\times\frac{y^2-y-2}{5ay}$ **b** $\frac{3}{x-3}+\frac{2x+8}{x^2-9}\times\frac{x^2+3x}{4x-16}$

c $\frac{5b}{2b+6}\div\frac{b^2}{b^2+b-6}-\frac{b}{b+1}$ **d** $\frac{x^2-8x+15}{5x^2+10x}\div\frac{x^2-9}{10x^2}\times\frac{x^2+5x+6}{2x-10}$

5 Simplify:

a $\frac{5}{x^2-4}-\frac{3}{x-2}-\frac{2}{x+2}$ **b** $\frac{2}{p^2+pq}+\frac{3}{pq-q^2}$ **c** $\frac{a}{a+b}-\frac{b}{a-b}+\frac{1}{a^2-b^2}$

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1.17 Substitution

Algebra is used for writing general formulas or rules, and we substitute numbers into these formulas to solve a problem.

EXAMPLE 19

a $V = \pi r^2 h$ is the formula for finding the volume of a cylinder with radius r and height h. Find V (correct to 1 decimal place) when $r = 2.1$ and $h = 8.7$.

b If $F = \frac{9C}{5} + 32$ is the formula for converting degrees Celsius (°C) into degrees Fahrenheit (°F), find F when $C = 25$.

Solution

a When $r = 2.1$, $h = 8.7$,

$$\begin{aligned} V &= \pi r^2 h \\ &= \pi(2.1)^2(8.7) \\ &= 120.533\ldots \\ &\approx 120.5 \end{aligned}$$

b When $C = 25$,

$$\begin{aligned} F &= \frac{9C}{5} + 32 \\ &= \frac{9(25)}{5} + 32 \\ &= 77 \end{aligned}$$

This means that 25°C is the same as 77°F.

Exercise 1.17 Substitution

1 Given $a = 3.1$ and $b = -2.3$ find, correct to 1 decimal place:

a ab **b** $3b$ **c** $5a^2$ **d** ab^3

e $(a + b)^2$ **f** $\sqrt{a - b}$ **g** $-b^2$

2 For the formula $T = a + (n - 1)d$, find T when $a = -4$, $n = 18$ and $d = 3$.

3 Given $y = mx + c$, the equation of a straight line, find y if $m = 3$, $x = -2$ and $c = -1$.

4 If $h = 100t - 5t^2$ is the height of a particle at time t, find h when $t = 5$.

5 Given vertical velocity $v = -gt$, find v when $g = 9.8$ and $t = 20$.

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6 If $y = 2^x + 3$ is the equation of a function, find y when $x = 1.3$, correct to 1 decimal place.

7 $S = 2\pi r(r + h)$ is the formula for the surface area of a cylinder. Find S when $r = 5$ and $h = 7$, correct to the nearest whole number.

8 $A = \pi r^2$ is the area of a circle with radius r. Find A when $r = 9.5$, correct to 3 significant figures.

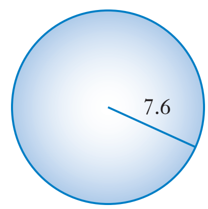
9 For the formula $u = ar^{n-1}$, find u if $a = 5$, $r = -2$ and $n = 4$.

10 Given $V = \frac{1}{3}lbh$ is the volume formula for a rectangular pyramid, find V if $l = 4.7$, $b = 5.1$ and $h = 6.5$.

11 The gradient of a straight line is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$. Find m if $x_1 = 3$, $x_2 = -1$, $y_1 = -2$ and $y_2 = 5$.

12 If $A = \frac{1}{2}h(a + b)$ gives the area of a trapezium, find A when $h = 7$, $a = 2.5$ and $b = 3.9$.

13 $V = \frac{4}{3}\pi r^3$ is the volume formula for a sphere with radius r.
Find V to 1 decimal place for a sphere with radius $r = 7.6$.



14 The velocity of an object at time t is given by the formula $v = u + at$.
Find v when $u = \frac{1}{4}$, $a = \frac{3}{5}$ and $t = \frac{5}{6}$.

15 Given $S = \frac{a}{1-r}$, find S if $a = 5$ and $r = \frac{2}{3}$. S is the sum to infinity of a geometric series.

16 $c = \sqrt{a^2 + b^2}$, according to Pythagoras' theorem. Find the value of c if $a = 6$ and $b = 8$.

17 Given $y = \sqrt{16 - x^2}$ is the equation of a semicircle, find the exact value of y when $x = 2$.

18 Find the value of E in the energy equation $E = mc^2$ if $m = 8.3$ and $c = 1.7$.

19 $A = P\left(1 + \frac{r}{100}\right)^n$ is the formula for finding compound interest. Find A correct to 2 decimal places when $P = 200$, $r = 12$ and $n = 5$.

20 If $S = \frac{a(r^n - 1)}{r - 1}$ is the sum of a geometric series, find S if $a = 3$, $r = 2$ and $n = 5$.

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1.18 Simplifying surds

An **irrational number** is a number that cannot be written as a ratio or fraction.

Surds such as $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{5}$ are special types of irrational numbers.

If a question involving surds asks for an exact answer, then leave it as a surd.

Properties of surds

$$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$$

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$$

$$\left(\sqrt{x}\right)^2 = \sqrt{x^2} = x \text{ for } x \geq 0$$

EXAMPLE 20

a Express $\sqrt{45}$ in simplest surd form.

b Simplify $3\sqrt{40}$.

c Write $5\sqrt{2}$ as a single surd.

Solution

a
$$\begin{aligned}\sqrt{45} &= \sqrt{9 \times 5}\\ &= \sqrt{9} \times \sqrt{5}\\ &= 3 \times \sqrt{5}\\ &= 3\sqrt{5}\end{aligned}$$

b
$$\begin{aligned}3\sqrt{40} &= 3 \times \sqrt{4} \times \sqrt{10}\\ &= 3 \times 2 \times \sqrt{10}\\ &= 6\sqrt{10}\end{aligned}$$

c
$$\begin{aligned}5\sqrt{2} &= \sqrt{25} \times \sqrt{2}\\ &= \sqrt{50}\end{aligned}$$

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Exercise 1.18 Simplifying surds

1 Express these surds in simplest surd form:

a	$\sqrt{12}$	**b**	$\sqrt{63}$	**c**	$\sqrt{24}$	**d**	$\sqrt{50}$	**e**	$\sqrt{72}$
f	$\sqrt{200}$	**g**	$\sqrt{48}$	**h**	$\sqrt{75}$	**i**	$\sqrt{32}$	**j**	$\sqrt{54}$
k	$\sqrt{112}$	**l**	$\sqrt{300}$	**m**	$\sqrt{128}$	**n**	$\sqrt{243}$	**o**	$\sqrt{245}$
p	$\sqrt{108}$	**q**	$\sqrt{99}$	**r**	$\sqrt{125}$				

2 Simplify:

a	$2\sqrt{27}$	**b**	$5\sqrt{80}$	**c**	$4\sqrt{98}$	**d**	$2\sqrt{28}$	**e**	$8\sqrt{20}$
f	$4\sqrt{56}$	**g**	$8\sqrt{405}$	**h**	$15\sqrt{8}$	**i**	$7\sqrt{40}$	**j**	$8\sqrt{45}$

3 Write as a single surd:

a	$3\sqrt{2}$	**b**	$2\sqrt{5}$	**c**	$4\sqrt{11}$	**d**	$8\sqrt{2}$	**e**	$5\sqrt{3}$
f	$4\sqrt{10}$	**g**	$3\sqrt{13}$	**h**	$7\sqrt{2}$	**i**	$11\sqrt{3}$	**j**	$12\sqrt{7}$

4 Evaluate x if:

a	$\sqrt{x}=3\sqrt{5}$	**b**	$2\sqrt{3}=\sqrt{x}$	**c**	$3\sqrt{7}=\sqrt{x}$	**d**	$5\sqrt{2}=\sqrt{x}$	**e**	$2\sqrt{11}=\sqrt{x}$
f	$\sqrt{x}=7\sqrt{3}$	**g**	$4\sqrt{19}=\sqrt{x}$	**h**	$\sqrt{x}=6\sqrt{23}$	**i**	$5\sqrt{31}=\sqrt{x}$	**j**	$\sqrt{x}=8\sqrt{15}$

1.19 Operations with surds

EXAMPLE 21

Simplify $\sqrt{3}-\sqrt{12}$.

Solution

First, change into like surds.

$$\begin{aligned}\sqrt{3}-\sqrt{12}&=\sqrt{3}-\sqrt{4}\times\sqrt{3}\\&=\sqrt{3}-2\sqrt{3}\\&=-\sqrt{3}\end{aligned}$$

Multiplication and division, as in algebra, are easier to do than adding and subtracting.

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EXAMPLE 22

Simplify:

a $4\sqrt{2}\times 5\sqrt{18}$ **b** $\dfrac{2\sqrt{14}}{4\sqrt{2}}$ **c** $\left(\sqrt{\dfrac{10}{3}}\right)^2$

Solution

a $$\begin{aligned}4\sqrt{2}\times 5\sqrt{18} &= 20\sqrt{36}\\ &= 20\times 6\\ &= 120\end{aligned}$$

b $$\begin{aligned}\frac{2\sqrt{14}}{4\sqrt{2}} &= \frac{2\times\sqrt{7}}{4}\\ &= \frac{\sqrt{7}}{2}\end{aligned}$$

c $$\begin{aligned}\left(\sqrt{\frac{10}{3}}\right)^2 &= \frac{10}{3}\\ &= 3\frac{1}{3}\end{aligned}$$

EXAMPLE 23

Expand and simplify:

a $3\sqrt{7}(2\sqrt{3}-3\sqrt{2})$ **b** $(\sqrt{2}+3\sqrt{5})(\sqrt{3}-\sqrt{2})$ **c** $(\sqrt{5}+2\sqrt{3})(\sqrt{5}-2\sqrt{3})$

Solution

a $$\begin{aligned}3\sqrt{7}(2\sqrt{3}-3\sqrt{2}) &= 3\sqrt{7}\times 2\sqrt{3}-3\sqrt{7}\times 3\sqrt{2}\\ &= 6\sqrt{21}-9\sqrt{14}\end{aligned}$$

b $$\begin{aligned}(\sqrt{2}+3\sqrt{5})(\sqrt{3}-\sqrt{2}) &= \sqrt{2}\times\sqrt{3}-\sqrt{2}\times\sqrt{2}+3\sqrt{5}\times\sqrt{3}-3\sqrt{5}\times\sqrt{2}\\ &= \sqrt{6}-2+3\sqrt{15}-3\sqrt{10}\end{aligned}$$

c Using the difference of 2 squares: $$\begin{aligned}(\sqrt{5}+2\sqrt{3})(\sqrt{5}-2\sqrt{3}) &= \left(\sqrt{5}\right)^2-\left(2\sqrt{3}\right)^2\\ &= 5-4\times 3\\ &= -7\end{aligned}$$

Exercise 1.19 Operations with surds

1 Simplify:

a $\sqrt{5}+2\sqrt{5}$ **b** $3\sqrt{2}-2\sqrt{2}$ **c** $\sqrt{3}+5\sqrt{3}$

d $7\sqrt{3}-4\sqrt{3}$ **e** $\sqrt{5}-4\sqrt{5}$ **f** $4\sqrt{6}-\sqrt{6}$

g $\sqrt{2}-8\sqrt{2}$ **h** $\sqrt{5}+4\sqrt{5}+3\sqrt{5}$ **i** $\sqrt{2}-2\sqrt{2}-3\sqrt{2}$

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j $\sqrt{5}+\sqrt{45}$ **k** $\sqrt{8}-\sqrt{2}$ **l** $\sqrt{3}+\sqrt{48}$

m $\sqrt{12}-\sqrt{27}$ **n** $\sqrt{50}-\sqrt{32}$ **o** $\sqrt{28}+\sqrt{63}$

p $2\sqrt{8}-\sqrt{18}$ **q** $3\sqrt{54}+2\sqrt{24}$ **r** $\sqrt{90}-5\sqrt{40}-2\sqrt{10}$

s $4\sqrt{48}+3\sqrt{147}+5\sqrt{12}$ **t** $3\sqrt{2}+\sqrt{8}-\sqrt{12}$ **u** $\sqrt{63}-\sqrt{28}-\sqrt{50}$

v $\sqrt{12}-\sqrt{45}-\sqrt{48}-\sqrt{5}$

2 Simplify:

a $\sqrt{7}\times\sqrt{3}$ **b** $\sqrt{3}\times\sqrt{5}$ **c** $\sqrt{2}\times3\sqrt{3}$

d $5\sqrt{7}\times2\sqrt{2}$ **e** $-3\sqrt{3}\times2\sqrt{2}$ **f** $5\sqrt{3}\times2\sqrt{3}$

g $-4\sqrt{5}\times3\sqrt{11}$ **h** $2\sqrt{7}\times\sqrt{7}$ **i** $2\sqrt{3}\times5\sqrt{12}$

j $\sqrt{6}\times\sqrt{2}$ **k** $\left(\sqrt{2}\right)^2$ **l** $\left(2\sqrt{7}\right)^2$

m $\sqrt{3}\times\sqrt{5}\times\sqrt{2}$ **n** $2\sqrt{3}\times\sqrt{7}\times-\sqrt{5}$ **o** $\sqrt{2}\times\sqrt{6}\times3\sqrt{3}$

3 Simplify:

a $\dfrac{4\sqrt{12}}{2\sqrt{2}}$ **b** $\dfrac{12\sqrt{18}}{3\sqrt{6}}$ **c** $\dfrac{5\sqrt{8}}{10\sqrt{2}}$ **d** $\dfrac{16\sqrt{2}}{2\sqrt{12}}$

e $\dfrac{10\sqrt{30}}{5\sqrt{10}}$ **f** $\dfrac{2\sqrt{2}}{6\sqrt{20}}$ **g** $\dfrac{4\sqrt{2}}{8\sqrt{10}}$ **h** $\dfrac{\sqrt{3}}{3\sqrt{15}}$

i $\dfrac{\sqrt{2}}{\sqrt{8}}$ **j** $\dfrac{3\sqrt{15}}{6\sqrt{10}}$ **k** $\dfrac{5\sqrt{12}}{5\sqrt{8}}$ **l** $\dfrac{15\sqrt{18}}{10\sqrt{10}}$

m $\dfrac{\sqrt{15}}{2\sqrt{6}}$ **n** $\left(\sqrt{\dfrac{2}{3}}\right)^2$ **o** $\left(\sqrt{\dfrac{5}{7}}\right)^2$

4 Expand and simplify:

a $\sqrt{2}\left(\sqrt{5}+\sqrt{3}\right)$ **b** $\sqrt{3}\left(2\sqrt{2}-\sqrt{5}\right)$ **c** $4\sqrt{3}\left(\sqrt{3}+2\sqrt{5}\right)$

d $\sqrt{7}\left(5\sqrt{2}-2\sqrt{3}\right)$ **e** $-\sqrt{3}\left(\sqrt{2}-4\sqrt{6}\right)$ **f** $\sqrt{3}\left(5\sqrt{11}+3\sqrt{7}\right)$

g $-3\sqrt{2}\left(\sqrt{2}+4\sqrt{3}\right)$ **h** $\sqrt{5}\left(\sqrt{5}-5\sqrt{3}\right)$ **i** $\sqrt{3}\left(\sqrt{12}+\sqrt{10}\right)$

j $2\sqrt{3}\left(\sqrt{18}+\sqrt{3}\right)$ **k** $-4\sqrt{2}\left(\sqrt{2}-3\sqrt{6}\right)$ **l** $-7\sqrt{5}\left(-3\sqrt{20}+2\sqrt{3}\right)$

m $10\sqrt{3}\left(\sqrt{2}-2\sqrt{12}\right)$ **n** $-\sqrt{2}\left(\sqrt{5}+2\right)$ **o** $2\sqrt{3}\left(2-\sqrt{12}\right)$

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5 Expand and simplify:

a $(\sqrt{2}+3)(\sqrt{5}+3\sqrt{3})$ **b** $(\sqrt{5}-\sqrt{2})(\sqrt{2}-\sqrt{7})$ **c** $(\sqrt{2}+5\sqrt{3})(2\sqrt{5}-3\sqrt{2})$

d $(3\sqrt{10}-2\sqrt{5})(4\sqrt{2}+6\sqrt{6})$ **e** $(2\sqrt{5}-7\sqrt{2})(\sqrt{5}-3\sqrt{2})$ **f** $(\sqrt{5}+6\sqrt{2})(3\sqrt{5}-\sqrt{3})$

g $(\sqrt{7}+\sqrt{3})(\sqrt{7}-\sqrt{3})$ **h** $(\sqrt{2}-\sqrt{3})(\sqrt{2}+\sqrt{3})$ **i** $(\sqrt{6}+3\sqrt{2})(\sqrt{6}-3\sqrt{2})$

j $(3\sqrt{5}+\sqrt{2})(3\sqrt{5}-\sqrt{2})$ **k** $(\sqrt{8}-\sqrt{5})(\sqrt{8}+\sqrt{5})$ **l** $(\sqrt{2}+9\sqrt{3})(\sqrt{2}-9\sqrt{3})$

m $(2\sqrt{11}+5\sqrt{2})(2\sqrt{11}-5\sqrt{2})$ **n** $(\sqrt{5}+\sqrt{2})^2$

o $(2\sqrt{2}-\sqrt{3})^2$ **p** $(3\sqrt{2}+\sqrt{7})^2$ **q** $(2\sqrt{3}+3\sqrt{5})^2$

r $(\sqrt{7}-2\sqrt{5})^2$ **s** $(2\sqrt{8}-3\sqrt{5})^2$ **t** $(3\sqrt{5}+2\sqrt{2})^2$

6 If $a=3\sqrt{2}$, simplify:

a a^2 **b** $2a^3$ **c** $(2a)^3$

d $(a+1)^2$ **e** $(a+3)(a-3)$

7 Evaluate a and b if:

a $(2\sqrt{5}+1)^2=a+\sqrt{b}$ **b** $(2\sqrt{2}-\sqrt{5})(\sqrt{2}-3\sqrt{5})=a+b\sqrt{10}$

8 Expand and simplify:

a $(\sqrt{a+3}-2)(\sqrt{a+3}+2)$ **b** $(\sqrt{p-1}-\sqrt{p})^2$

9 Evaluate $(2\sqrt{7}-\sqrt{3})(2\sqrt{7}+\sqrt{3})$.

10 Simplify $(2\sqrt{x}+\sqrt{y})(\sqrt{x}-3\sqrt{y})$.

11 If $(2\sqrt{3}-\sqrt{5})^2=a-\sqrt{b}$, evaluate a and b.

12 Evaluate a and b if $(7\sqrt{2}-3)^2=a+b\sqrt{2}$.

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1.20 Rationalising the denominator

Rationalising the denominator of a fractional surd means writing it with a rational number (not a surd) in the denominator. For example, after rationalising the denominator $\frac{3}{\sqrt{5}}$ becomes $\frac{3\sqrt{5}}{5}$.

WS Rationalising the denominator

WS Surds

To rationalise the denominator, multiply top and bottom by the same surd as in the denominator:

Rationalising the denominator

$$\frac{a}{\sqrt{b}} \times \frac{\sqrt{b}}{\sqrt{b}} = \frac{a\sqrt{b}}{b}$$

EXAMPLE 24

Rationalise the denominator of $\frac{2}{5\sqrt{3}}$.

Solution

$$\begin{aligned}\frac{2}{5\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} &= \frac{2\sqrt{3}}{5\sqrt{9}} \\ &= \frac{2\sqrt{3}}{5 \times 3} \\ &= \frac{2\sqrt{3}}{15}\end{aligned}$$

When there is a binomial denominator, we use the difference of 2 squares to rationalise it.

Rationalising a binomial denominator

To rationalise the denominator of $\frac{b}{\sqrt{c}+\sqrt{d}}$, multiply by $\frac{\sqrt{c}-\sqrt{d}}{\sqrt{c}-\sqrt{d}}$.

To rationalise the denominator of $\frac{b}{\sqrt{c}-\sqrt{d}}$, multiply by $\frac{\sqrt{c}+\sqrt{d}}{\sqrt{c}+\sqrt{d}}$.

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EXAMPLE 25

a Write with a rational denominator:

i $\dfrac{\sqrt{5}}{\sqrt{2}-3}$ **ii** $\dfrac{2\sqrt{3}+\sqrt{5}}{\sqrt{3}+4\sqrt{2}}$

b Evaluate a and b if $\dfrac{3\sqrt{3}}{\sqrt{3}-\sqrt{2}}=a+\sqrt{b}$.

c Evaluate $\dfrac{2}{\sqrt{3}+2}+\dfrac{\sqrt{5}}{\sqrt{3}-2}$ as a fraction with rational denominator.

Solution

a **i**

$$\begin{aligned}\frac{\sqrt{5}}{\sqrt{2}-3}\times\frac{\sqrt{2}+3}{\sqrt{2}+3}&=\frac{\sqrt{5}\left(\sqrt{2}+3\right)}{\left(\sqrt{2}\right)^2-3^2}\\&=\frac{\sqrt{10}+3\sqrt{5}}{2-9}\\&=-\frac{\sqrt{10}+3\sqrt{5}}{7}\end{aligned}$$

ii

$$\begin{aligned}\frac{2\sqrt{3}+\sqrt{5}}{\sqrt{3}+4\sqrt{2}}\times\frac{\sqrt{3}-4\sqrt{2}}{\sqrt{3}-4\sqrt{2}}&=\frac{\left(2\sqrt{3}+\sqrt{5}\right)\left(\sqrt{3}-4\sqrt{2}\right)}{\left(\sqrt{3}\right)^2-\left(4\sqrt{2}\right)^2}\\&=\frac{2\times3-8\sqrt{6}+\sqrt{15}-4\sqrt{10}}{3-16\times2}\\&=\frac{6-8\sqrt{6}+\sqrt{15}-4\sqrt{10}}{-29}\\&=\frac{-6+8\sqrt{6}-\sqrt{15}+4\sqrt{10}}{29}\end{aligned}$$

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b $\dfrac{3\sqrt{3}}{\sqrt{3}-\sqrt{2}}\times\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}}=\dfrac{3\sqrt{3}\left(\sqrt{3}+\sqrt{2}\right)}{\left(\sqrt{3}-\sqrt{2}\right)\left(\sqrt{3}+\sqrt{2}\right)}$

$$=\frac{3\sqrt{9}+3\sqrt{6}}{\left(\sqrt{3}\right)^2-\left(\sqrt{2}\right)^2}$$

$$=\frac{3\times3+3\sqrt{6}}{3-2}$$

$$=\frac{9+3\sqrt{6}}{1}$$

$$=9+3\sqrt{6}$$

$$=9+\sqrt{9}\times\sqrt{6}$$

$$=9+\sqrt{54}$$

So $a = 9$ and $b = 54$.

c $\dfrac{2}{\sqrt{3}+2}+\dfrac{\sqrt{5}}{\sqrt{3}-2}=\dfrac{2\left(\sqrt{3}-2\right)+\sqrt{5}\left(\sqrt{3}+2\right)}{\left(\sqrt{3}+2\right)\left(\sqrt{3}-2\right)}$

$$=\frac{2\sqrt{3}-4+\sqrt{15}+2\sqrt{5}}{\left(\sqrt{3}\right)^2-2^2}$$

$$=\frac{2\sqrt{3}-4+\sqrt{15}+2\sqrt{5}}{3-4}$$

$$=\frac{2\sqrt{3}-4+\sqrt{15}+2\sqrt{5}}{-1}$$

$$=-2\sqrt{3}+4-\sqrt{15}-2\sqrt{5}$$

Exercise 1.20 Rationalising the denominator

1 Express with a rational denominator:

a $\dfrac{1}{\sqrt{7}}$ **b** $\dfrac{\sqrt{3}}{2\sqrt{2}}$ **c** $\dfrac{2\sqrt{3}}{\sqrt{5}}$ **d** $\dfrac{6\sqrt{7}}{5\sqrt{2}}$

e $\dfrac{1+\sqrt{2}}{\sqrt{3}}$ **f** $\dfrac{\sqrt{6}-5}{\sqrt{2}}$ **g** $\dfrac{\sqrt{5}+2\sqrt{2}}{\sqrt{5}}$ **h** $\dfrac{3\sqrt{2}-4}{2\sqrt{7}}$

i $\dfrac{8+3\sqrt{2}}{4\sqrt{5}}$ **j** $\dfrac{4\sqrt{3}-2\sqrt{2}}{7\sqrt{5}}$

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2 Express with a rational denominator:

a $\dfrac{4}{\sqrt{3}+\sqrt{2}}$ **b** $\dfrac{\sqrt{3}}{\sqrt{2}-7}$ **c** $\dfrac{2\sqrt{3}}{\sqrt{5}+2\sqrt{6}}$

d $\dfrac{\sqrt{3}-4}{\sqrt{3}+4}$ **e** $\dfrac{\sqrt{2}+5}{\sqrt{3}-\sqrt{2}}$ **f** $\dfrac{3\sqrt{3}+\sqrt{2}}{2\sqrt{5}+3\sqrt{2}}$

3 Express as a single fraction with a rational denominator:

a $\dfrac{1}{\sqrt{2}+1}+\dfrac{1}{\sqrt{2}-1}$ **b** $\dfrac{\sqrt{2}}{\sqrt{2}-\sqrt{3}}-\dfrac{3}{\sqrt{2}+\sqrt{3}}$

c $t+\dfrac{1}{t}$ where $t=\sqrt{3}-2$ **d** $z^2-\dfrac{1}{z^2}$ where $z=1+\sqrt{2}$

e $\dfrac{\sqrt{2}+3}{\sqrt{2}}+\dfrac{1}{\sqrt{3}}$ **f** $\dfrac{\sqrt{3}}{\sqrt{2}+3}+\dfrac{\sqrt{2}}{\sqrt{3}}$

g $\dfrac{\sqrt{5}}{\sqrt{6}+2}-\dfrac{2}{5\sqrt{3}}$ **h** $\dfrac{\sqrt{2}+7}{4+\sqrt{3}}-\dfrac{\sqrt{2}}{4-\sqrt{3}}$

i $\dfrac{\sqrt{5}-\sqrt{2}}{\sqrt{3}-\sqrt{2}}-\dfrac{2+\sqrt{3}}{\sqrt{3}+1}$

4 Find a and b if:

a $\dfrac{3}{2\sqrt{5}}=\dfrac{\sqrt{a}}{b}$ **b** $\dfrac{\sqrt{3}}{4\sqrt{2}}=\dfrac{a\sqrt{6}}{b}$ **c** $\dfrac{2}{\sqrt{5}+1}=a+b\sqrt{5}$

d $\dfrac{2\sqrt{7}}{\sqrt{7}-4}=a+b\sqrt{7}$ **e** $\dfrac{\sqrt{2}+3}{\sqrt{2}-1}=a+\sqrt{b}$

5 Show that $\dfrac{\sqrt{2}-1}{\sqrt{2}+1}+\dfrac{4}{\sqrt{2}}$ is rational.

6 If $x=\sqrt{3}+2$, simplify:

a $x+\dfrac{1}{x}$ **b** $x^2+\dfrac{1}{x^2}$ **c** $\left(x+\dfrac{1}{x}\right)^2$

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1. TEST YOURSELF

Practice quiz

For Questions **1** to **8**, select the correct answer **A**, **B**, **C** or **D**.

1 Rationalise the denominator of $\frac{\sqrt{3}}{2\sqrt{7}}$ (there may be more than one answer).

A $\frac{\sqrt{21}}{28}$ **B** $\frac{2\sqrt{21}}{28}$ **C** $\frac{\sqrt{21}}{14}$ **D** $\frac{\sqrt{21}}{7}$

2 Simplify $\frac{x-3}{5}-\frac{x+1}{4}$.

A $\frac{-(x+7)}{20}$ **B** $\frac{x+7}{20}$ **C** $\frac{x+17}{20}$ **D** $\frac{-(x+17)}{20}$

3 Factorise x^3-4x^2-x+4 (there may be more than one answer).

A $(x^2-1)(x-4)$ **B** $(x^2+1)(x-4)$

C $x^2(x-4)$ **D** $(x-4)(x+1)(x-1)$

4 Simplify $3\sqrt{2}+2\sqrt{98}$.

A $5\sqrt{2}$ **B** $5\sqrt{10}$ **C** $17\sqrt{2}$ **D** $10\sqrt{2}$

5 Simplify $\frac{3}{x^2-4}+\frac{2}{x-2}-\frac{1}{x+2}$.

A $\frac{x+5}{(x+2)(x-2)}$ **B** $\frac{x+1}{(x+2)(x-2)}$ **C** $\frac{x+9}{(x+2)(x-2)}$ **D** $\frac{x-3}{(x+2)(x-2)}$

6 Simplify $5ab-2a^2-7ab-3a^2$.

A $2ab+a^2$ **B** $-2ab-5a^2$ **C** $-13a^3b$ **D** $-2ab+5a^2$

7 Simplify $\sqrt{\frac{80}{27}}$.

A $\frac{4\sqrt{5}}{3\sqrt{3}}$ **B** $\frac{4\sqrt{5}}{9\sqrt{3}}$ **C** $\frac{8\sqrt{5}}{9\sqrt{3}}$ **D** $\frac{8\sqrt{5}}{3\sqrt{3}}$

8 Expand and simplify $(3x-2y)^2$.

A $3x^2-12xy-2y^2$ **B** $9x^2-12xy-4y^2$

C $3x^2-6xy+2y^2$ **D** $9x^2-12xy+4y^2$

9 Evaluate as a fraction:

a 7^{-2} **b** 5^{-1} **c** $9^{-\frac{1}{2}}$

10 Simplify:

a $x^5 \times x^7 \div x^3$ **b** $(5y^3)^2$ **c** $\dfrac{(a^5)^4 b^7}{a^9 b}$ **d** $\left(\dfrac{2x^6}{3}\right)^3$ **e** $\left(\dfrac{ab^4}{a^5 b^6}\right)^0$

11 Evaluate:

a $36^{\frac{1}{2}}$ **b** 4^{-3} as fraction **c** $8^{\frac{2}{3}}$

d $49^{-\frac{1}{2}}$ as a fraction **e** $16^{\frac{1}{4}}$ **f** $(-3)^0$

12 Simplify:

a $a^{14} \div a^9$ **b** $(x^5y^3)^6$ **c** $p^6 \times p^5 \div p^2$

d $(2b^9)^4$ **e** $\dfrac{(2x^7)^3 y^2}{x^{10} y}$

13 Write in index form:

a $\sqrt{n}$ **b** $\dfrac{1}{x^5}$ **c** $\dfrac{1}{x+y}$ **d** $\sqrt[4]{x+1}$ **e** $\sqrt[7]{a+b}$

f $\dfrac{2}{x}$ **g** $\dfrac{1}{2x^3}$ **h** $\sqrt[3]{x^4}$ **i** $\sqrt[7]{(5x+3)^9}$ **j** $\dfrac{1}{\sqrt[4]{m^3}}$

14 Write without fractional or negative indices:

a a^{-5} **b** $n^{\frac{1}{4}}$ **c** $(x+1)^{\frac{1}{2}}$ **d** $(x-y)^{-1}$ **e** $(4t-7)^{-4}$

f $(a+b)^{\frac{1}{5}}$ **g** $x^{-\frac{1}{3}}$ **h** $b^{\frac{3}{4}}$ **i** $(2x+3)^{\frac{4}{3}}$ **j** $x^{-\frac{3}{2}}$

15 Evaluate a^2b^4 when $a = \dfrac{9}{25}$ and $b = 1\dfrac{2}{3}$.

16 If $a = \left(\dfrac{1}{3}\right)^4$ and $b = \dfrac{3}{4}$, evaluate ab^3 as a fraction.

17 Write in index form:

a $\sqrt{x}$ **b** $\dfrac{1}{y}$ **c** $\sqrt[6]{x+3}$ **d** $\dfrac{1}{(2x-3)^{11}}$ **e** $\sqrt[3]{y^7}$

18 Write without the negative index:

a x^{-3} **b** $(2a+5)^{-1}$ **c** $\left(\dfrac{a}{b}\right)^{-5}$

19 Simplify:

a $5y - 7y$ **b** $\dfrac{3a+12}{3}$ **c** $-2k^3 \times 3k^2$ **d** $\dfrac{x}{3} + \dfrac{y}{5}$

e $4a - 3b - a - 5b$ **f** $\sqrt{8} + \sqrt{32}$ **g** $3\sqrt{5} - \sqrt{20} + \sqrt{45}$

ISBN 9780170413152

20 Factorise:

a $x^2 - 36$ **b** $a^2 + 2a - 3$ **c** $4ab^2 - 8ab$

d $5y - 15 + xy - 3x$ **e** $4n - 2p + 6$

21 Expand and simplify:

a $b + 3(b - 2)$ **b** $(2x - 1)(x + 3)$ **c** $5(m + 3) - (m - 2)$

d $(4x - 3)^2$ **e** $(p - 5)(p + 5)$ **f** $7 - 2(a + 4) - 5a$

g $\sqrt{3}\left(2\sqrt{2}-5\right)$ **h** $\left(3+\sqrt{7}\right)\left(\sqrt{3}-2\right)$

22 Simplify:

a $\dfrac{4a-12}{5b^3}\times\dfrac{10b}{a^2-9}$ **b** $\dfrac{5m+10}{m^2-m-2}\div\dfrac{m^2-4}{3m+3}$

23 The volume of a cube is $V = s^3$. Evaluate V when $s = 5.4$.

24 a Expand and simplify $\left(2\sqrt{5}+\sqrt{3}\right)\left(2\sqrt{5}-\sqrt{3}\right)$.

b Rationalise the denominator of $\dfrac{3\sqrt{3}}{2\sqrt{5}+\sqrt{3}}$.

25 Simplify $\dfrac{3}{x-2}+\dfrac{1}{x+3}-\dfrac{2}{x^2+x-6}$.

26 If $a = 4$, $b = -3$ and $c = -2$, find the value of:

a ab^2 **b** $a - bc$ **c** $\sqrt{a}$ **d** $(bc)^3$ **e** $c(2a + 3b)$

27 Simplify:

a $\dfrac{3\sqrt{12}}{6\sqrt{15}}$ **b** $\dfrac{4\sqrt{32}}{2\sqrt{2}}$

28 The formula for the distance an object falls is given by $d = 5t^2$. Find d when $t = 1.5$.

29 Rationalise the denominator of:

a $\dfrac{2}{5\sqrt{3}}$ **b** $\dfrac{1+\sqrt{3}}{\sqrt{2}}$

30 Expand and simplify:

a $\left(3\sqrt{2}-4\right)\left(\sqrt{3}-\sqrt{2}\right)$ **b** $\left(\sqrt{7}+2\right)^2$

31 Factorise fully:

a $3x^2 - 27$ **b** $6x^2 - 12x - 18$ **c** $5y^2 - 30y + 45$

32 Simplify:

a $\dfrac{3x^4y}{9xy^5}$ **b** $\dfrac{5}{15x-5}$

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33 Simplify:

a $\left(3\sqrt{11}\right)^2$ **b** $\left(2\sqrt{3}\right)^3$

34 Expand and simplify:

a $(a+b)(a-b)$ **b** $(a+b)^2$

35 Factorise:

a $a^2 - 2ab + b^2$ **b** $a^2 - b^2$

36 If $x = \sqrt{3} + 1$, simplify $x + \frac{1}{x}$ and give your answer with a rational denominator.

37 Simplify:

a $\frac{4}{a} + \frac{3}{b}$ **b** $\frac{x-3}{2} - \frac{x-2}{5}$

38 Simplify $\frac{3}{\sqrt{5}+2} - \frac{\sqrt{2}}{2\sqrt{2}-1}$, writing your answer with a rational denominator.

39 Simplify:

a $3\sqrt{8}$ **b** $-2\sqrt{2} \times 4\sqrt{3}$ **c** $\sqrt{108} - \sqrt{48}$ **d** $\frac{8\sqrt{6}}{2\sqrt{18}}$

e $5a \times -3b \times -2a$ **f** $\frac{2m^3n}{6m^2n^5}$ **g** $3x - 2y - x - y$

40 Expand and simplify:

a $2\sqrt{2}\left(\sqrt{3}+\sqrt{2}\right)$ **b** $\left(5\sqrt{7}-3\sqrt{5}\right)\left(2\sqrt{2}-\sqrt{3}\right)$ **c** $\left(3+\sqrt{2}\right)\left(3-\sqrt{2}\right)$

d $\left(4\sqrt{3}-\sqrt{5}\right)\left(4\sqrt{3}+\sqrt{5}\right)$ **e** $\left(3\sqrt{7}-\sqrt{2}\right)^2$

41 Rationalise the denominator of:

a $\frac{3}{\sqrt{7}}$ **b** $\frac{\sqrt{2}}{5\sqrt{3}}$ **c** $\frac{2}{\sqrt{5}-1}$ **d** $\frac{2\sqrt{2}}{3\sqrt{2}+\sqrt{3}}$ **e** $\frac{\sqrt{5}+\sqrt{2}}{4\sqrt{5}-3\sqrt{3}}$

42 Simplify:

a $\frac{3x}{5} - \frac{x-2}{2}$ **b** $\frac{a+2}{7} + \frac{2a-3}{3}$ **c** $\frac{1}{x^2-1} - \frac{2}{x+1}$

d $\frac{4}{k^2+2k-3} + \frac{1}{k+3}$ **e** $\frac{\sqrt{3}}{\sqrt{2}+\sqrt{5}} - \frac{5}{\sqrt{3}-\sqrt{2}}$

43 Evaluate n if:

a $\sqrt{108} - \sqrt{12} = \sqrt{n}$ **b** $\sqrt{112} + \sqrt{7} = \sqrt{n}$ **c** $2\sqrt{8} + \sqrt{200} = \sqrt{n}$

d $4\sqrt{147} + 3\sqrt{75} = \sqrt{n}$ **e** $2\sqrt{245} + \frac{\sqrt{180}}{2} = \sqrt{n}$

ISBN 9780170413152

1. CHALLENGE EXERCISE

1 Write $64^{-\frac{2}{3}}$ as a rational number.

2 Show that $2(2^k - 1) + 2^{k+1} = 2(2^{k+1} - 1)$.

3 Find the value of $\frac{a}{b^3c^2}$ in index form if $a = \left(\frac{2}{5}\right)^4$, $b = \left(-\frac{1}{3}\right)^3$ and $c = \left(\frac{3}{5}\right)^2$.

4 Expand and simplify:

a $4ab(a - 2b) - 2a^2(b - 3a)$ **b** $(y^2 - 2)(y^2 + 2)$ **c** $(2x - 5)^3$

5 Find the value of $x + y$ with rational denominator if $x = \sqrt{3} + 1$ and $y = \frac{1}{2\sqrt{5} - 3}$.

6 Simplify $\frac{2\sqrt{3}}{7\sqrt{6} - \sqrt{54}}$.

7 Factorise:

a $(x + 4)^2 + 5(x + 4)$ **b** $x^4 - x^2y - 6y^2$ **c** $a^2b - 2a^2 - 4b + 8$

8 Simplify $\frac{2xy + 2x - 6 - 6y}{4x^2 - 16x + 12}$.

9 Simplify $\frac{(a+1)^3}{a^2 - 1}$.

10 Factorise $\frac{4}{x^2} - \frac{a^2}{b^2}$.

11 **a** Expand $(2x - 1)^3$.

b Hence, or otherwise, simplify $\frac{6x^2 + 5x - 4}{8x^3 - 12x^2 + 6x - 1}$.

12 If $V = \pi r^2 h$ is the volume of a cylinder, find the exact value of r when $V = 9$ and $h = 16$.

13 If $s = u + \frac{1}{2}at^2$, find the exact value of s when $u = 2$, $a = \sqrt{3}$ and $t = 2\sqrt{3}$.

14 Expand and simplify, and write in index form:

a $\left(\sqrt{x} + x\right)^2$ **b** $\left(\sqrt[3]{a} + \sqrt[3]{b}\right)\left(\sqrt[3]{a} - \sqrt[3]{b}\right)$

c $\left(p + \frac{1}{\sqrt{p}}\right)^2$ **d** $\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2$

15 Find the value of $\frac{a^3b^2}{c^2}$ if $a = \left(\frac{3}{4}\right)^2$, $b = \left(\frac{2}{3}\right)^3$ and $c = \left(\frac{1}{2}\right)^4$.

ISBN 9780170413152