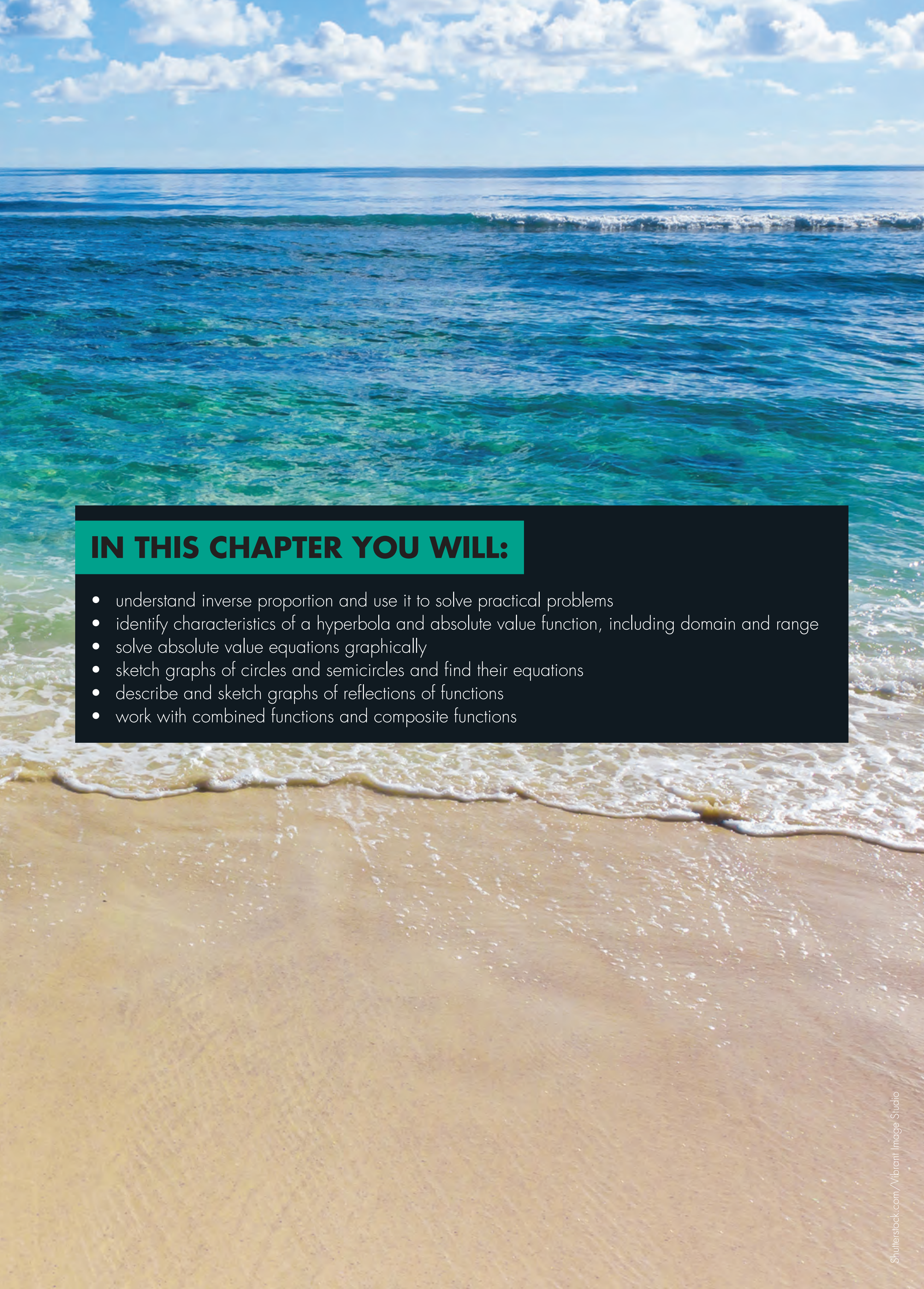
5. FURTHER FUNCTIONS

In this chapter, we look at functions and relations that are not polynomials, including the hyperbola, absolute value, circles and semicircles. We will also study reflections and relationships between functions, including combined functions and composite functions.

CHAPTER OUTLINE



IN THIS CHAPTER YOU WILL:

- understand inverse proportion and use it to solve practical problems
- identify characteristics of a hyperbola and absolute value function, including domain and range
- solve absolute value equations graphically
- sketch graphs of circles and semicircles and find their equations
- describe and sketch graphs of reflections of functions
- work with combined functions and composite functions

TERMINOLOGY

asymptote: A line that a curve approaches but doesn't touch

composite function: A function of a function, where the output of one function becomes the input of a second function, written as $f(g(x))$. For example, if $f(x) = x^2$ and $g(x) = 3x + 1$ then $f(g(x)) = (3x + 1)^2$

continuous function: A function whose graph is smooth and does not have gaps or breaks

discontinuous function: A function whose graph has a gap or break in it; for example, $f(x) = \frac{1}{x}$, whose graph is a hyperbola

hyperbola: The graph of the function $y = \frac{k}{x}$, which is made up of 2 separate curves

inverse variation: A relationship between 2 variables such that as one variable increases the other variable decreases, or as one variable decreases the other variable increases. One variable is a multiple of the reciprocal of the other, with equation $y = \frac{k}{x}$. Also called **inverse proportion**

5.01 The hyperbola

WS The hyperbola

Graphing hyperbolas

WS Graphing $y = \frac{10}{x}$

Inverse variation

We looked at direct variation and the equation $y = kx$ in Chapter 3, *Functions*. When one variable is in **inverse variation** (or **inverse proportion**) with another variable, one is a constant multiple of the **reciprocal** of the other. This means that as one variable increases, the other decreases and when one decreases, the other increases.

For example:

- The more slices you cut a pizza into, the smaller the size of each slice
- The more workers there are on a project, the less time it takes to complete
- The fewer people sharing a house, the higher the rent each person pays.

Inverse variation

If variables x and y are in inverse variation, we can write the equation $y = \frac{k}{x}$ where k is called the **constant of variation**.

EXAMPLE 1

a Building a shed in 12 hours requires 3 builders. If the number of builders, N, is in inverse variation to the amount of time, t hours:

i find the equation for N in terms of t

ii find the number of builders it would take to build the shed in 9 hours

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iii find how long it would take 2 builders to build the shed

iv graph the equation for N after completing the table below.

t	1	2	3	4	5	6	7	8	9
N									

b The faster a car travels, the less time it takes to travel a certain distance. It takes the car 2 hours to travel this distance at a speed of 80 km/h. If the time taken, t hours, is in inverse proportion to the speed s km/h, then:

i find the equation for t in terms of s

ii find the time it would take if travelling at 100 km/h

iii find the speed at which the trip would take $2\frac{1}{2}$ hours

iv graph the equation.

Solution

a **i** For inverse variation, the equation is in the form $N = \frac{k}{t}$.

Substitute $t = 12$, $N = 3$ to find the value of k:

$$3 = \frac{k}{12}$$

$$36 = k$$

$$\therefore N = \frac{36}{t}$$

ii Substitute $t = 9$.

$$N = \frac{36}{9}$$

$$= 4$$

So it takes 4 builders to build the shed in 9 hours.

iii Substitute $N = 2$.

$$2 = \frac{36}{t}$$

$$2t = 36$$

$$t = 18$$

So it takes 18 hours for 2 builders to build the shed.

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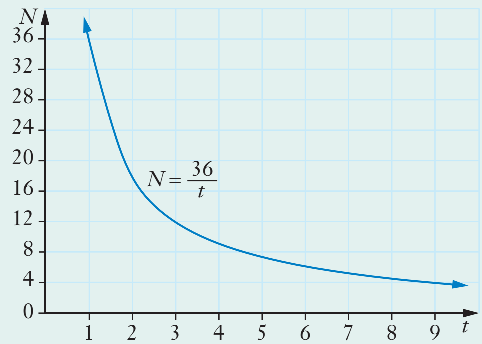
iv $N = \dfrac{36}{t}$

Completing a table of values:

t	1	2	3	4	5	6	7	8	9
N	36	18	12	9	7.2	6	5.1	4.5	4



b **i** For inverse proportion, the equation is in the form $t = \dfrac{k}{s}$.

Substitute $s = 80$, $t = 2$ to find k.

$$2 = \frac{k}{80}$$

$$k = 160$$

$$\therefore \quad t = \frac{160}{s}$$

ii Substitute $s = 100$.

$$t = \frac{160}{100}$$

$$= 1.6 \text{ hours}$$

$$= 1 \text{ h } 36 \text{ min}$$

So the car takes 1 h 36 min to travel the distance if travelling at 100 km/h.

iii Substitute $t = 2.5$.

$$2.5 = \frac{160}{s}$$

$$2.5s = 160$$

$$s = \frac{160}{2.5}$$

$$= 64$$

So the car travels at 64 km/h if the trip takes $2\frac{1}{2}$ hours.

iv To graph this function, complete a table of values for $t = \dfrac{160}{s}$.

s	20	40	60	80	100	120
t	8	4	2.667	2	1.6	1.333

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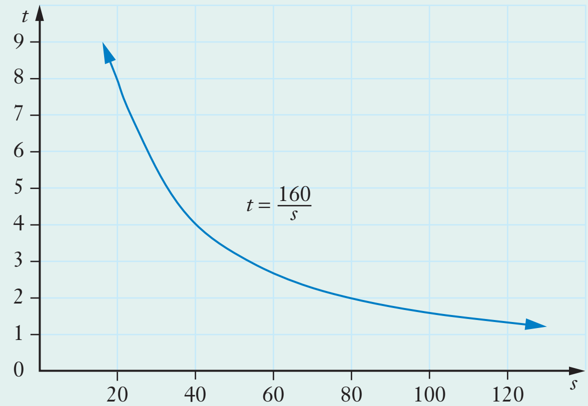


The graph of the function $y = \frac{k}{x}$ is a **hyperbola**.

Hyperbolas

A hyperbola is the graph of a function of the form $y = \frac{k}{x}$.

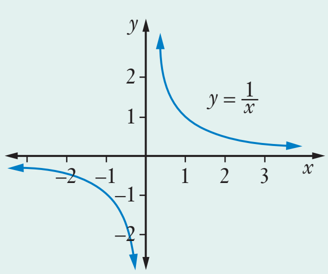
EXAMPLE 2

Sketch the graph of $y = \frac{1}{x}$. What is the domain and range?

Solution

x	-3	-2	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	1	2	3
y	$-\frac{1}{3}$	$-\frac{1}{2}$	-1	-2	-4	–	4	2	1	$\frac{1}{2}$	$\frac{1}{3}$

When $x = 0$ the value of y is undefined.



Domain: x can be any real number except 0.
$(-\infty, 0) \cup (0, \infty)$

Range: y can be any real number except 0.
$(-\infty, 0) \cup (0, \infty)$

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CLASS DISCUSSION

LIMITS OF THE HYPERBOLA

What happens to the graph as x becomes closer to 0? What happens as x becomes very large in both positive and negative directions? The value of y is never 0. Why?

Continuity

Most functions have graphs that are smooth unbroken curves (or lines). They are called **continuous functions**. However, some functions have discontinuities, meaning that their graphs have gaps or breaks. These are called **discontinuous functions**.

The hyperbola is discontinuous because there is a gap in the graph and it has two separate parts. The graph of $y = \frac{1}{x}$ also does not touch the x- or y-axes, but it gets closer and closer to them. We call the x- and y-axes **asymptotes**: lines that the curve approaches but never touches.

To find the shape of the graph close to the asymptotes or as $x \to \pm\infty$, we can check points nearby.

EXAMPLE 3

Find the domain and range of $f(x) = \frac{3}{x-3}$ and sketch the graph of the function.

Solution

To find the domain, we notice that $x - 3 \neq 0$. So $x \neq 3$.

Domain $(-\infty, 3) \cup (3, \infty)$

Also y cannot be zero: $y \neq 0$.

Range $(-\infty, 0) \cup (0, \infty)$

The lines $x = 3$ and $y = 0$ (the x-axis) are the asymptotes of the hyperbola.

To find the limiting behaviour of the graph, look at what is happening as $x \to \pm\infty$.

As x increases and approaches ∞, $\frac{3}{x-3}$ becomes closer to 0 and is positive.
Substitute large values of x into the function, for example, $x = 1000$.

As $x \to \infty$, $y \to 0^+$ (as x approaches infinity, y approaches 0 from above, the positive side).

Similarly, as x decreases and approaches $-\infty$, y becomes closer to 0 and is negative.
Substitute $x = -1000$, for example.

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As $x \to -\infty, y \to 0^-$ (as x approaches negative infinity, y approaches 0 from below, the negative side).

To see the behaviour of the function near the asymptote $x = 3$ we can test values either side.

LHS: When $x = 2.999$,

$$\frac{3}{2.999-3} = -3000$$
$$< 0$$

As $x \to 3^-, y \to -\infty$

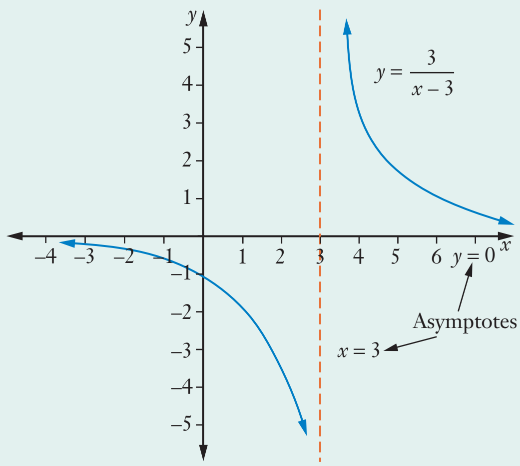
RHS: When $x = 3.001$,

$$\frac{3}{3.001-3} = 3000$$
$$> 0$$

As $x \to 3^+, y \to \infty$

For y-intercept, $x = 0$:

$$y = \frac{3}{0-3}$$
$$= -1$$



EXAMPLE 4

Sketch the graph of $y = -\dfrac{1}{2x+4}$.

Solution

To find the domain, notice that:

$2x + 4 \neq 0$

$2x \neq -4$

$x \neq -2$

Domain $(-\infty, -2) \cup (-2, \infty)$

For the range, $y \neq 0$.

Range $(-\infty, 0) \cup (0, \infty)$

So there are asymptotes at $x = -2$ and $y = 0$.

Limiting behaviour:

Substitute, say, $x = 5000$ and $x = -5000$.

As $x \to \infty, y \to 0^-$.

As $x \to -\infty, y \to 0^+$.

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To see the shape of the graph near the asymptote $x = -2$, we can test values either side.

LHS: When $x = -2.0001$,

$$y = -\frac{1}{2(-2.0001)+4} = 5000$$
$$> 0$$

As $x \to 2^-, y \to \infty$

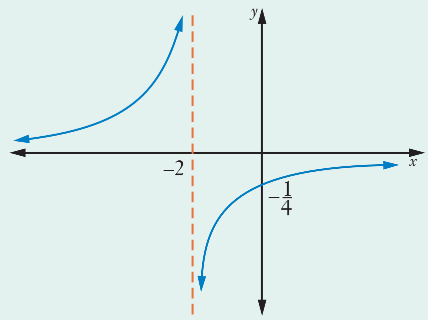
RHS: When $x = -1.9999$,

$$y = -\frac{1}{2(-1.9999)+4} = -5000$$
$$< 0$$

As $x \to 2^+, y \to -\infty$

For y-intercept, $x = 0$

$$y = -\frac{1}{2(0)+4}$$
$$= -\frac{1}{4}$$



The hyperbola

The hyperbola $y = \frac{k}{bx+c}$ is a discontinuous function with 2 parts, separated by vertical and horizontal asymptotes.

Exercise 5.01 The hyperbola

1 The diameter of a balloon varies inversely with the thickness of the rubber. The diameter of the balloon is 80 mm when the rubber is 2 mm thick.

- **a** Find an equation for the diameter D in terms of the thickness x.
- **b** Find the diameter when the thickness is 0.8 mm.
- **c** Find the thickness correct to one decimal place when the diameter is 115.3 mm.
- **d** Sketch the graph showing this information.

2 The more boxes a factory produces, the less it costs to produce each box. When 128 boxes are produced, it costs \$2 per box.

- **a** Write an equation for the cost c to produce each box when manufacturing n boxes.
- **b** Find the cost of each box when 100 boxes are produced.
- **c** Find how many boxes must be produced for the cost for each box to be 50 cents.
- **d** Sketch the graph of this information.

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3 For each function:

i state the domain and range

ii find the y-intercept if it exists

iii sketch the graph.

a $y = \dfrac{2}{x}$ **b** $y = -\dfrac{1}{x}$ **c** $f(x) = \dfrac{1}{x+1}$

d $f(x) = \dfrac{3}{x-2}$ **e** $y = \dfrac{1}{3x+6}$ **f** $f(x) = -\dfrac{2}{x-3}$

g $f(x) = \dfrac{4}{x-1}$ **h** $y = -\dfrac{2}{x+1}$ **i** $f(x) = \dfrac{2}{6x-3}$

4 Show that $f(x) = \dfrac{2}{x}$ is an odd function.

5 a Is the hyperbola $y = -\dfrac{2}{x+1}$:

i a function? **ii** even, odd or neither? **iii** continuous?

b What are the equations of the asymptotes?

c State its domain and range.

5.02 Absolute value functions

WS Absolute value functions

An absolute value function is an example of a piecewise function with 2 sections. We were introduced to absolute value in Chapter 3, *Functions*.

The absolute value function

$$|x| = \begin{cases} x & \text{if } x \text{ ° } 0 \\ -x & \text{if } x < 0 \end{cases}$$

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EXAMPLE 5

Sketch the graph of $y = |x|$ and state its domain and range.

Solution

$y = |x|$ gives the piecewise function:

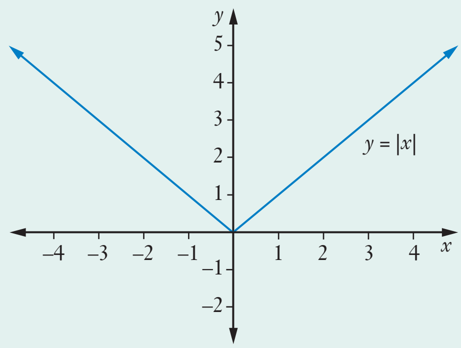
$$y = \begin{cases} x & \text{for } x \ {}^\circ\ 0 \\ -x & \text{for } x < 0 \end{cases}$$

We can draw $y = x$ for $x \geq 0$ and $y = -x$ for $x < 0$ on the same set of axes.

From the graph, notice that x can be any real number while $y \geq 0$.

Domain $(-\infty, \infty)$

Range $[0, \infty)$



Absolute value graphs

EXAMPLE 6

a Sketch the graph of $f(x) = |x| - 1$ and state its domain and range.

b Sketch the graph of $y = |x + 2|$.

Solution

a Using the definition of absolute value:

$$y = \begin{cases} x - 1 & \text{for } x \geq 0 \\ -x - 1 & \text{for } x < 0 \end{cases}$$

Draw $y = x - 1$ for $x \geq 0$ and $y = -x - 1$ for $x < 0$.

For x-intercepts, $y = 0$:

$y = x - 1$	$y = -x - 1$
$0 = x - 1$	$0 = -x - 1$
$1 = x$	$x = -1$

For y-intercept, $x = 0$:

$$\begin{aligned} y &= x - 1 \text{ for } x = 0 \\ &= 0 - 1 \\ &= -1 \end{aligned}$$

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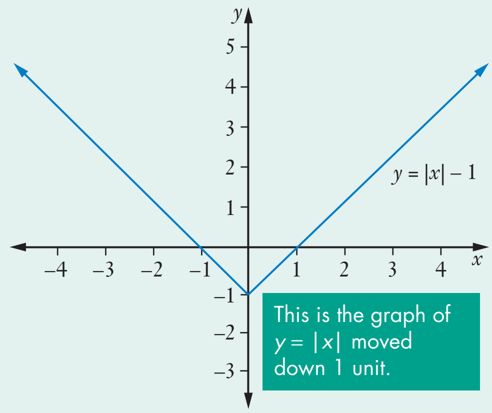
From the graph, notice that x can be any real number while $y \geq -1$.

Domain: $(-\infty, \infty)$

Range: $[-1, \infty)$



b Using the definition of absolute value:

$$y = \begin{cases} x+2 & \text{for } x+2 \geq 0 \\ -(x+2) & \text{for } x+2 < 0 \end{cases}$$

Simplifying this gives:

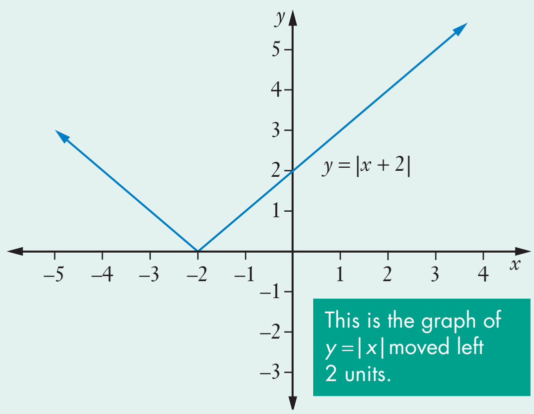
$$y = \begin{cases} x+2 & \text{for } x \geq -2 \\ -x-2 & \text{for } x < -2 \end{cases}$$

For x-intercepts, $y = 0$:

$$\begin{aligned} y &= x+2 \qquad & y &= -x-2 \\ 0 &= x+2 & 0 &= -x-2 \\ -2 &= x & x &= -2 \end{aligned}$$

For y-intercept, $x = 0$:

$$\begin{aligned} y &= x+2 \text{ for } x = 0 \\ &= 0+2 \\ &= 2 \end{aligned}$$



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INVESTIGATION

TRANSFORMATIONS OF THE ABSOLUTE VALUE FUNCTION

Use a graphics calculator or graphing software to explore each absolute value graph.

1 $y = \|x\|$	**2** $y = 2\|x\|$	**3** $y = 3\|x\|$
4 $y = -\|x\|$	**5** $y = -2\|x\|$	**6** $y = \|x\| + 1$
7 $y = \|x\| + 2$	**8** $y = \|x\| - 1$	**9** $y = \|x\| - 2$
10 $y = \|x + 1\|$	**11** $y = \|x + 2\|$	**12** $y = \|x + 3\|$
13 $y = \|x - 1\|$	**14** $y = \|x - 2\|$	**15** $y = \|x - 3\|$

Are graphs that involve absolute value always functions? Can you find an example of one that is not a function?

Are any of them odd or even? Are they continuous? Could you predict what the graph $y = 2|x - 7|$ would look like?

Equations involving absolute values

We learned how to solve equations involving absolute values using algebra in Chapter 2, *Equations and inequalities*. We can also solve these equations graphically.

EXAMPLE 7

Solve $|2x - 1| = 3$ graphically.

Solution

Sketch the graphs of $y = |2x - 1|$ and $y = 3$ on the same number plane.

$$y = \begin{cases} 2x - 1 & \text{for } 2x - 1 \geq 0 \\ -(2x - 1) & \text{for } 2x - 1 < 0 \end{cases}$$

Simplifying this gives:

$$y = \begin{cases} 2x - 1 & \text{for } x \geq \frac{1}{2} \\ -2x + 1 & \text{for } x < \frac{1}{2} \end{cases}$$

For x-intercepts, $y = 0$:

$$\begin{aligned} y &= 2x - 1 \\ 0 &= 2x - 1 \\ 1 &= 2x \\ \frac{1}{2} &= x \end{aligned} \qquad \begin{aligned} y &= -2x + 1 \\ 0 &= -2x + 1 \\ 2x &= 1 \\ x &= \frac{1}{2} \end{aligned}$$

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For y-intercept, $x = 0$:

$y = -2x + 1$ for $x = 0$

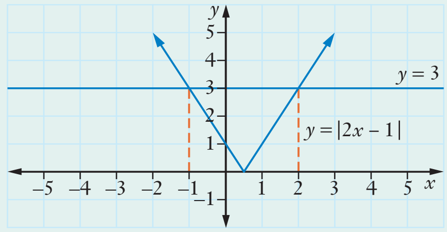
$= -2(0) + 1$

$= 1$

The graph of $y = 3$ is a horizontal line through 3 on the y-axis.

The solutions of $|2x - 1| = 3$ are the values of x at the point of intersection of the graphs.

$x = -1, 2$.



We can check that our solutions are correct by substituting them back into the equation.

Exercise 5.02 Absolute value functions

1 Find the x- and y-intercepts of the graph of each function.

a $f(x) = |x| + 7$ **b** $f(x) = |x| - 2$ **c** $y = 5|x|$
d $f(x) = -|x| + 3$ **e** $y = |x + 6|$ **f** $f(x) = |3x - 2|$
g $y = |5x + 4|$ **h** $y = |7x - 1|$ **i** $f(x) = |2x| + 9$

2 Sketch the graph of each function.

a $y = |x|$ **b** $f(x) = |x| + 1$ **c** $f(x) = |x| - 3$
d $y = 2|x|$ **e** $f(x) = -|x|$ **f** $y = |x + 1|$
g $f(x) = -|x - 1|$ **h** $y = |2x - 3|$ **i** $f(x) = |3x| + 1$

3 Find the domain and range of each function.

a $y = |x - 1|$ **b** $f(x) = |x| - 8$ **c** $f(x) = |2x + 5|$
d $y = 2|x| - 3$ **e** $f(x) = -|x - 3|$

4 Solve each equation graphically.

a $|x| = 3$ **b** $|x + 2| = 1$ **c** $|x - 3| = 0$
d $|2x - 3| = 1$ **e** $|2x + 3| = 11$ **f** $|5b - 2| = 8$
g $|3x + 1| = 2$ **h** $5 = |2x + 1|$ **i** $0 = |6t - 3|$

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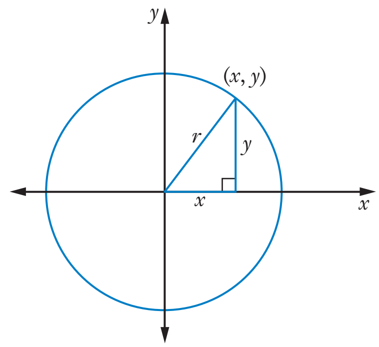
5.03 Circles and semicircles

WS
Equations of circles

The circle is not a function. It does not pass the vertical line test.

Circle with centre (0, 0)

We can use Pythagoras' theorem to find the equation of a circle, using a general point (x, y) on a circle with centre $(0, 0)$ and radius r.



$$c^2 = a^2 + b^2$$

$$\therefore \quad r^2 = x^2 + y^2$$

Equation of a circle with centre (0, 0)

The equation of a circle with centre $(0, 0)$ and radius r is $x^2 + y^2 = r^2$.

EXAMPLE 8

a Sketch the graph of $x^2 + y^2 = 4$.

b Why is it not a function?

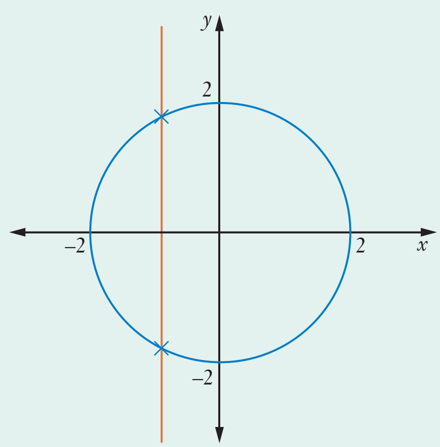
c State its domain and range.

Solution

a The equation is in the form $x^2 + y^2 = r^2$ where $r^2 = 4$.

Radius $r = \sqrt{4} = 2$

This is a circle with radius 2 and centre $(0, 0)$.



b The circle is not a function because a vertical line will cut the graph in more than one place.

c The x values for this graph lie between -2 and 2 and the y values also lie between -2 and 2.

Domain: $[-2, 2]$

Range: $[-2, 2]$

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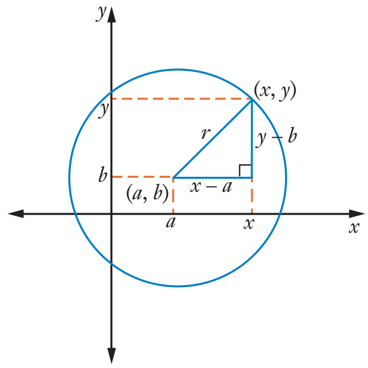
Circle with centre (a, b)

We can use Pythagoras' theorem to find the equation of a circle using a general point (x, y) on a circle with centre (a, b) and radius r.

The smaller sides of the triangle are $x - a$ and $y - b$ and the hypotenuse is r, the radius.

$c^2 = a^2 + b^2$

$r^2 = (x - a)^2 + (y - b)^2$



Equation of a circle with centre (a, b)

The equation of a circle with centre (a, b) and radius r is $(x - a)^2 + (y - b)^2 = r^2$.

EXAMPLE 9

a **i** Sketch the graph of the circle $(x - 1)^2 + (y + 2)^2 = 4$.

ii State its domain and range.

b Find the equation of a circle with radius 3 and centre $(-2, 1)$ in expanded form.

c Find the centre and radius of the circle with equation $x^2 + 2x + y^2 - 6y - 6 = 0$.

Solution

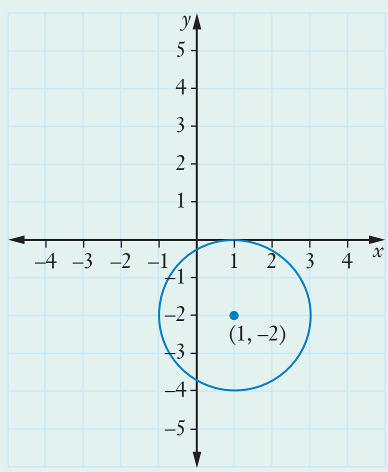
a **i** The equation is in the form $(x - a)^2 + (y - b)^2 = r^2$.

$(x - 1)^2 + (y + 2)^2 = 4$

$(x - 1)^2 + (y - (-2))^2 = 2^2$

So $a = 1$, $b = -2$ and $r = 2$.

This is a circle with centre $(1, -2)$ and radius 2.



ii From the graph, we can see that all x values lie between -1 and 3 and all y values lie between -4 and 0.

Domain: $[-1, 3]$

Range: $[-4, 0]$

b Centre is $(-2, 1)$ so $a = -2$ and $b = 1$.

Radius is 3 so $r = 3$.

$$(x-a)^2 + (y-b)^2 = r^2$$
$$(x-(-2))^2 + (y-1)^2 = 3^2$$
$$(x+2)^2 + (y-1)^2 = 9$$

Expanding:

$$x^2 + 4x + 4 + y^2 - 2y + 1 = 9$$
$$x^2 + 4x + y^2 - 2y - 4 = 0$$

c The equation of a circle is $(x-a)^2 + (y-b)^2 = r^2$.

We need to complete the square to put the equation into this form.

To complete the square on $x^2 + 2x$, we add $\left(\frac{2}{2}\right)^2 = 1$.

To complete the square on $y^2 - 6y$, we add $\left(\frac{6}{2}\right)^2 = 9$.

$$x^2 + 2x + y^2 - 6y - 6 = 0$$
$$x^2 + 2x + y^2 - 6y = 6$$
$$x^2 + 2x + 1 + y^2 - 6y + 9 = 6 + 1 + 9$$
$$(x+1)^2 + (y-3)^2 = 16$$
$$(x-(-1))^2 + (y-3)^2 = 4^2$$

This is in the form $(x-a)^2 + (y-b)^2 = r^2$ where $a = -1$, $b = 3$ and $r = 4$.

So it is a circle with centre $(-1, 3)$ and radius 4 units.

Semicircles

By rearranging the equation of a circle, we can find the equations of 2 semicircles.

$$x^2 + y^2 = r^2$$
$$y^2 = r^2 - x^2$$
$$y = \pm\sqrt{r^2 - x^2}$$

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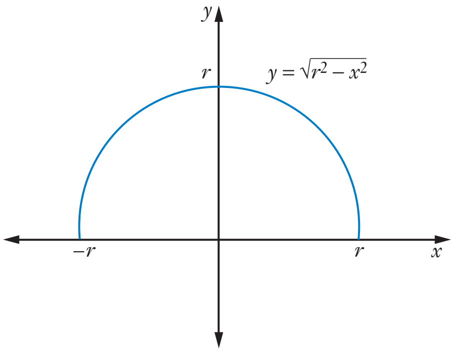
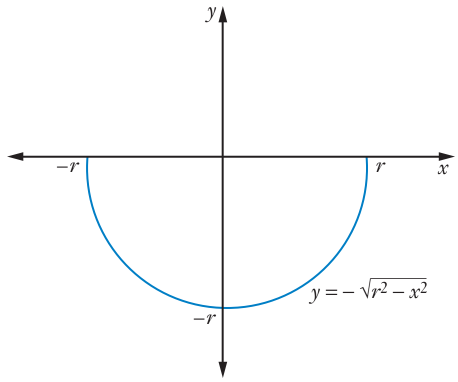
This gives 2 separate functions:





$y = \sqrt{r^2 - x^2}$ is the semicircle above the x-axis since $y \geq 0$.

$y = -\sqrt{r^2 - x^2}$ is the semicircle below the x-axis since $y \leq 0$.

Equations of a semicircle with centre (0, 0)

The equation of a semicircle above the x-axis with centre (0, 0) and radius r is

$$y = \sqrt{r^2 - x^2}.$$

The equation of a semicircle below the x-axis with centre (0, 0) and radius r is

$$y = -\sqrt{r^2 - x^2}.$$

EXAMPLE 10

Sketch the graph of each function and state the domain and range.

a $f(x) = \sqrt{9 - x^2}$ **b** $y = -\sqrt{4 - x^2}$

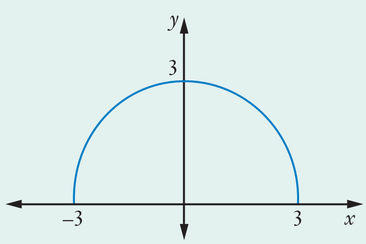
Solution

a This is in the form $f(x) = \sqrt{r^2 - x^2}$ where $r^2 = 9$, so $r = 3$.

It is a semicircle above the x-axis with centre (0, 0) and radius 3.

Domain: [–3, 3]

Range: [0, 3]



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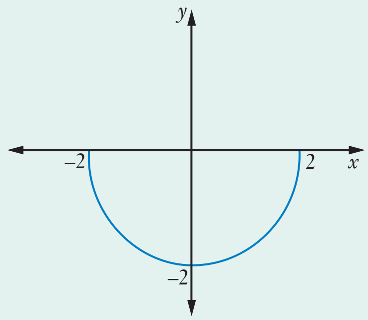
b This is in the form $y = -\sqrt{r^2 - x^2}$ where $r^2 = 4$, so $r = 2$.

It is a semicircle below the x-axis with centre (0, 0) and radius 2.

Domain: [–2, 2]

Range: [–2, 0]



Exercise 5.03 Circles and semicircles

1 For each equation:

i sketch the graph

ii state the domain and range.

a $x^2 + y^2 = 9$

b $x^2 + y^2 - 16 = 0$

c $(x - 2)^2 + (y - 1)^2 = 4$

d $(x + 1)^2 + y^2 = 9$

e $(x + 2)^2 + (y - 1)^2 = 1$

2 For each semicircle:

i state whether it is above or below the x-axis

ii sketch the graph

iii state the domain and range.

a $y = -\sqrt{25 - x^2}$

b $y = \sqrt{1 - x^2}$

c $y = \sqrt{36 - x^2}$

d $y = -\sqrt{64 - x^2}$

e $y = -\sqrt{7 - x^2}$

3 Find the radius and the centre of each circle.

a $x^2 + y^2 = 100$

b $x^2 + y^2 = 5$

c $(x - 4)^2 + (y - 5)^2 = 16$

d $(x - 5)^2 + (y + 6)^2 = 49$

e $x^2 + (y - 3)^2 = 81$

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4 Find the equation of each circle in expanded form.

a centre (0, 0) and radius 4
b centre (3, 2) and radius 5
c centre (–1, 5) and radius 3
d centre (2, 3) and radius 6
e centre (–4, 2) and radius 5
f centre (0, –2) and radius 1
g centre (4, 2) and radius 7
h centre (–3, –4) and radius 9
i centre (–2, 0) and radius $\sqrt{5}$
j centre (–4, –7) and radius $\sqrt{3}$

5 Find the radius and the centre of each circle.

a $x^2 - 4x + y^2 - 2y - 4 = 0$
b $x^2 + 8x + y^2 - 4y - 5 = 0$
c $x^2 + y^2 - 2y = 0$
d $x^2 - 10x + y^2 + 6y - 2 = 0$
e $x^2 + 2x + y^2 - 2y + 1 = 0$
f $x^2 - 12x + y^2 = 0$
g $x^2 + 6x + y^2 - 8y = 0$
h $x^2 + 20x + y^2 - 4y + 40 = 0$
i $x^2 - 14x + y^2 + 2y + 25 = 0$
j $x^2 + 2x + y^2 + 4y - 5 = 0$

6 Find the centre and radius of the circle with equation:

a $x^2 - 6x + y^2 + 2y - 6 = 0$
b $x^2 - 4x + y^2 - 10y + 4 = 0$
c $x^2 + 2x + y^2 + 12y - 12 = 0$
d $x^2 - 8x + y^2 - 14y + 1 = 0$

7 Sketch the circle whose equation is given by $x^2 + 4x + y^2 - 2y + 1 = 0$.

5.04 Reflections of functions

The graph of $y = -f(x)$

WS Advanced graphs

WS Matching graphs (Advanced)

EXAMPLE 11

For each function, sketch the graph of $y = f(x)$ and $y = -f(x)$ on the same number plane.

a $f(x) = x^2 - 2x$ b $f(x) = x^3$

Solution

a $f(x) = x^2 - 2x$ is a concave upwards parabola.

For x-intercepts: $f(x) = 0$

$x^2 - 2x = 0$

$x(x - 2) = 0$

$x = 0, 2$

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For y-intercept, $x = 0$:

$f(0) = 0^2 - 2(0) = 0$

Axis of symmetry at $x = 1$ (halfway between 0 and 2):

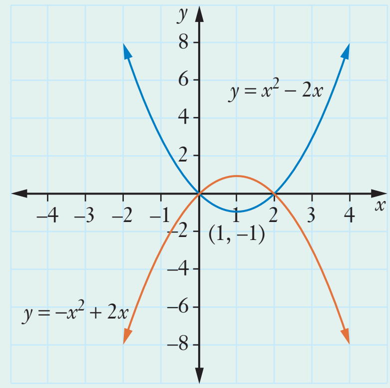
$f(1) = 1^2 - 2(1) = -1$

Minimum turning point at $(1, -1)$.

$y = -f(x)$

$= -(x^2 - 2x)$

$= -x^2 + 2x$



$y = -x^2 + 2x$ is a concave downwards parabola also with x-intercepts 0, 2 with y-intercept 0 and axis of symmetry at $x = 1$.

$f(1) = -1^2 + 2(1) = 1$

Maximum turning point at $(1, 1)$.

Draw both graphs on the same set of axes.

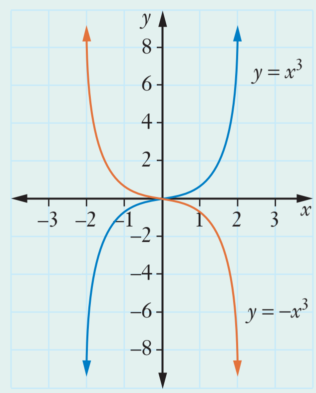
b $f(x) = x^3$ is a cubic curve with a point of inflection at $(0, 0)$.

x	−3	−2	−1	0	1	2	3
y	−27	−8	−1	0	1	8	27

$y = -f(x)$

$= -x^3$

x	−3	−2	−1	0	1	2	3
y	27	8	1	0	−1	−8	−27



$y = -f(x)$ changes the sign of the y values of the original function: from positive to negative, or negative to positive. On the number plane, this means reflecting the graph in the x-axis.

The graph of $y = -f(x)$

The graph of $y = -f(x)$ is a reflection of the graph of $y = f(x)$ in the x-axis.

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The graph of $y = f(-x)$

We have already seen that some functions are even or odd by finding $f(-x)$. We can see the relationship between $f(x)$ and $f(-x)$ by drawing their graphs.

EXAMPLE 12

For each function, sketch the graph of $y = f(x)$ and $y = f(-x)$ on a number plane.

a $f(x) = x^3 + 1$ **b** $f(x) = \dfrac{1}{x-2}$

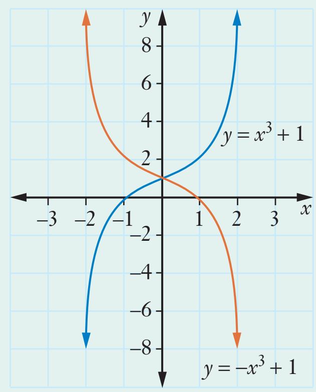
Solution

a $f(x) = x^3 + 1$ is a cubic curve with point of inflection at (0, 1).

x	–3	–2	–1	0	1	2	3
y	–26	–7	0	1	2	9	28

$$y = f(-x)$$
$$= (-x)^3 + 1$$
$$= -x^3 + 1$$

x	–3	–2	–1	0	1	2	3
y	28	9	2	1	0	–7	–26



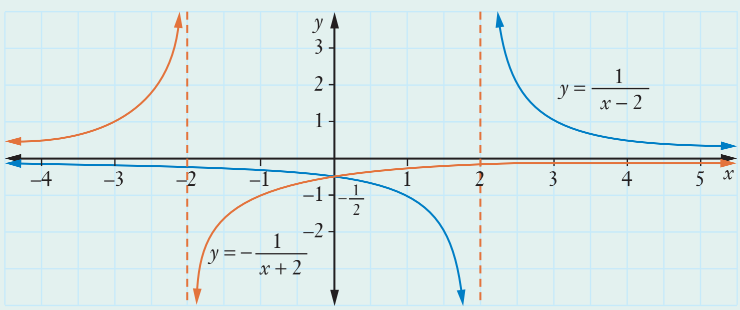
Draw $y = x^3 + 1$ and $y = -x^3 + 1$ on the same set of axes.

b $f(x) = \dfrac{1}{x-2}$ is a hyperbola with asymptotes at $x = 2$ and $y = 0$.

y-intercept, $x = 0$:

$$f(0) = \frac{1}{0-2} = -\frac{1}{2}$$

$$y = f(-x)$$
$$= \frac{1}{-x-2}$$
$$= -\frac{1}{x+2}$$



Asymptotes at $x = -2$ and $y = 0$.

y-intercept, $x = 0$:

$$f(0) = -\frac{1}{0+2} = -\frac{1}{2}$$

Draw $y = \dfrac{1}{x-2}$ and $y = -\dfrac{1}{x+2}$ on the same set of axes.

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$y = f(-x)$ changes the sign of the x value of the original function: from positive to negative, or negative to positive. On the number plane, this means reflecting the graph in the y-axis.

The graph of $y = f(-x)$

The graph of $y = f(-x)$ is a reflection of the graph of $y = f(x)$ in the y-axis.

The graph of $y = -f(-x)$

EXAMPLE 13

For each function, sketch the graph of $y = f(x)$ and $y = -f(-x)$ on the same number plane.

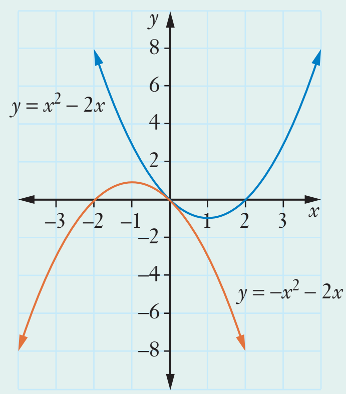
a $f(x) = x^2 - 2x$ **b** $f(x) = \dfrac{2}{x+1}$

Solution

a From Example **12a**, $f(x) = x^2 - 2x$ is a concave upwards parabola with x-intercepts 0, 2.

$$\begin{aligned} y &= -f(-x) \\ &= -\left([-x]^2 - 2[-x]\right) \\ &= -\left(x^2 + 2x\right) \\ &= -x^2 - 2x \end{aligned}$$

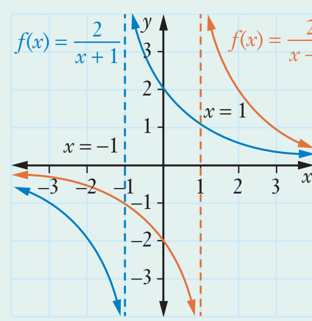
A concave downwards parabola with x-intercepts 0, −2.



b $f(x) = \dfrac{2}{x+1}$

This is a hyperbola with asymptotes at $x = -1$, $y = 0$ and y-intercept $f(0) = 2$.

$$\begin{aligned} y &= -f(-x) \\ &= -\frac{2}{-x+1} \\ &= \frac{2}{x-1} \end{aligned}$$



This is a hyperbola with asymptotes at $x = 1$, $y = 0$ and y-intercept $f(0) = -2$.

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The graph of $y = -f(-x)$

$y = -f(-x)$ is a reflection of the graph of $y = f(x)$ in both the x- and y-axes.

CLASS DISCUSSION

REFLECTIONS OF FUNCTIONS

Use a graphics calculator or graphing software to draw the graphs of different functions $y = f(x)$ together with:

1 $y = -f(x)$

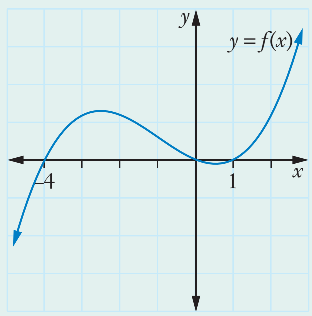
2 $y = f(-x)$

3 $y = -f(-x)$.

Are any of these functions the same as $y = f(x)$ if $f(x)$ is an even or odd function? Why?

EXAMPLE 14

The graph of $y = f(x)$ is shown below.

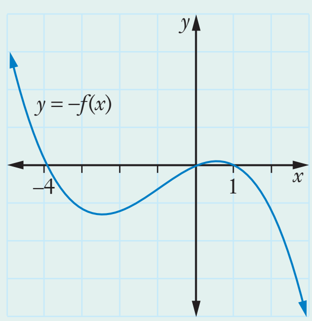


Sketch the graph of:

a $y = -f(x)$ **b** $y = f(-x)$ **c** $y = -f(-x)$

Solution

a $y = -f(x)$ is a reflection in the x-axis.



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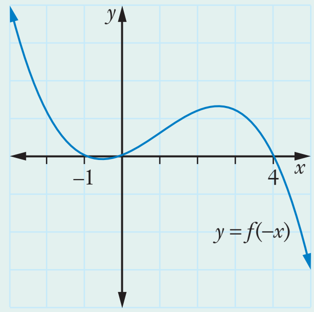
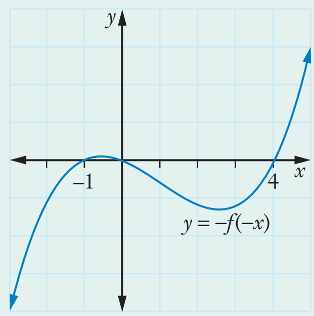
b $y = f(-x)$ is a reflection in the y-axis.



c $y = -f(-x)$ is a reflection in both the x-and y-axes. Using the graph from **a** that has been reflected in the x-axis and reflecting it in the y-axis gives the graph below.



Exercise 5.04 Reflections of functions

1 For each function, find the equation of:

i $y = -f(x)$ **ii** $y = f(-x)$ **iii** $y = -f(-x)$

a $f(x) = x^2 - 2$ **b** $f(x) = (x + 1)^3$ **c** $y = 5x - 3$

d $y = |2x + 5|$ **e** $f(x) = \dfrac{1}{x-1}$

2 Describe the type of reflection that each function has on $y = f(x)$.

a $y = -f(x)$ **b** $y = f(-x)$ **c** $y = -f(-x)$

3 Sketch the graphs of the function $f(x) = (x - 1)^2$ and $y = -f(-x)$ on the same number plane.

4 Sketch the graphs of the function $f(x) = 1 - x^3$ and $y = -f(x)$ on the same number plane.

5 For the function $f(x) = x^2 + 2x$, sketch the graph of:

a $y = f(x)$ **b** $y = -f(x)$ **c** $y = f(-x)$ **d** $y = -f(-x)$

6 **a** Show that $f(x) = 2x^2$ is an even function.

b Find the equation of:

i $y = f(-x)$ **ii** $y = -f(x)$

c Sketch the graph of $y = -f(-x)$.

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7 **a** Show that $f(x) = -x^3$ is an odd function.

b Find the equation of:

i $y = -f(x)$ **ii** $y = -f(-x)$

c Sketch the graph of $y = f(-x)$.

8 **a** Find the x- and y-intercepts of the graph of $f(x) = x^3 - 7x^2 + 12x$ and sketch the graph.

b Sketch the graph of:

i $y = f(-x)$ **ii** $y = -f(x)$ **iii** $y = -f(-x)$

5.05 Combined and composite functions

Sometimes we use different operations to combine 2 different functions.

WS

Circles and composite functions

EXAMPLE 15

For $f(x) = 2x^2 - x + 1$ and $g(x) = x^3 - 2$, write each combined function below as a polynomial and find its degree and constant term.

a $y = f(x) + g(x)$ **b** $y = f(x) - g(x)$ **c** $y = f(x)g(x)$

Solution

a $y = f(x) + g(x)$

$= 2x^2 - x + 1 + x^3 - 2$

$= x^3 + 2x^2 - x - 1$

This polynomial has degree 3 and constant term -1.

b $y = f(x) - g(x)$

$= 2x^2 - x + 1 - (x^3 - 2)$

$= 2x^2 - x + 1 - x^3 + 2$

$= -x^3 + 2x^2 - x + 3$

This polynomial has degree 3 and constant term 3.

c $y = f(x)g(x)$

$= (2x^2 - x + 1)(x^3 - 2)$

$= 2x^5 - 4x^2 - x^4 + 2x + x^3 - 2$

$= 2x^5 - x^4 + x^3 - 4x^2 + 2x - 2$

This polynomial has degree 5 and constant term -2.

We could also find the degree by multiplying just the 2 leading terms: $2x^2 \times x^3 = 2x^5$ and find the constant term by multiplying just the 2 constant terms: $1 \times (-2) = (-2)$.

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EXAMPLE 16

a Find the domain and range of each function below given $f(x) = x^2 - x - 2$ and $g(x) = x - 2$.

i $y = f(x) + g(x)$ **ii** $y = f(x) - g(x)$ **iii** $y = f(x)g(x)$

b Find the domain of $y = \dfrac{f(x)}{g(x)}$ if $f(x) = x^3 + 1$ and $g(x) = x^2 - x - 6$.

Solution

a **i** $y = f(x) + g(x)$

$= x^2 - x - 2 + x - 2$

$= x^2 - 4$

This is a quadratic function with a minimum turning point at $(0, -4)$.

Domain $(-\infty, \infty)$, range $[-4, \infty)$

ii $y = f(x) - g(x)$

$= x^2 - x - 2 - (x - 2)$

$= x^2 - x - 2 - x + 2$

$= x^2 - 2x$

This is a quadratic function with x-intercepts 0, 2.

Axis of symmetry: $x = 1$

Minimum value:

$f(1) = 1^2 - 2(1)$

$= -1$

Domain $(-\infty, \infty)$, range $[-1, \infty)$

iii $y = f(x)g(x)$

$= (x^2 - x - 2)(x - 2)$

$= x^3 - 2x^2 - x^2 + 2x - 2x + 4$

$= x^3 - 3x^2 + 4$

This is a cubic function.

Domain $(-\infty, \infty)$, range $(-\infty, \infty)$

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b $y = \dfrac{f(x)}{g(x)}$

$= \dfrac{x^3+1}{x^2-x-6}$

For domain: $x^2 - x - 6 \neq 0$

$(x-3)(x+2) \neq 0$

$x \neq 3, -2$

So domain is $(-\infty, -2) \cup (-2, 3) \cup (3, \infty)$

Composite functions

A **composite function** $f(g(x))$ is a relationship between functions where the output of one function $g(x)$ becomes the input of a second function $f(x)$.

EXAMPLE 17

a Find the composite function $f(g(x))$ given:

i $f(x) = x^2$ and $g(x) = 2x - 5$

ii $f(x) = x^3$ and $g(x) = x^2 + 3$

iii $f(x) = 5x - 3$ and $g(x) = x^3 + 2$

b Given $f(x) = 5x + 2$ and $g(x) = \dfrac{1}{x}$, find:

i $f(g(x))$ **ii** $g(f(x))$

c Find the domain and range of $f(g(x))$ given $f(x) = \sqrt{x}$ and $g(x) = 9 - x^2$.

Solution

a **i** $f(g(x)) = (2x-5)^2$

$= 4x^2 - 20x + 25$

ii $f(g(x)) = (x^2+3)^3$

$= (x^2+3)(x^2+3)^2$

$= (x^2+3)(x^4+6x^2+9)$

$= x^6 + 9x^4 + 27x^2 + 27$

iii $f(g(x)) = 5(x^3+2) - 3$

$= 5x^3 + 10 - 3$

$= 5x^3 + 7$

b **i** $f(g(x)) = 5\left(\dfrac{1}{x}\right) + 2$

$= \dfrac{5}{x} + 2$

ii $g(f(x)) = \dfrac{1}{5x+2}$

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c $f(g(x)) = \sqrt{9 - x^2}$

This is a semicircle above the x-axis with centre (0, 0) and radius 3.

Domain $[-3, 3]$, range $[0, 3]$

Exercise 5.05 Combined and composite functions

1 For each pair of functions, find the combined function:

i $y = f(x) + g(x)$ **ii** $y = f(x) - g(x)$

iii $y = f(x)g(x)$ **iv** $y = \dfrac{f(x)}{g(x)}$

a $f(x) = 4x + 1$ and $g(x) = 2x^2 + x$

b $f(x) = x^4 + 5x - 4$ and $g(x) = x^3 + 5$

c $f(x) = x^2 + 3$ and $g(x) = 5x^2 - 7x - 2$

d $f(x) = 3x^2 + 2x - 1$ and $g(x) = x^2 - x + 5$

e $f(x) = 4x^5 + 7$ and $g(x) = 3x - 4$

2 For each pair of functions, find the degree of:

i $f(x) + g(x)$ **ii** $f(x) - g(x)$ **iii** $f(x)g(x)$ without expanding

a $f(x) = 2x + 1$ and $g(x) = 5x - 7$ **b** $f(x) = x^2$ and $g(x) = 3x + 4$

c $f(x) = (x - 3)^2$ and $g(x) = x^2 - 6x + 1$ **d** $f(x) = 2x^3$ and $g(x) = x - 2$

3 For each pair of functions, find the constant term of:

i $f(x) + g(x)$ **ii** $f(x) - g(x)$ **iii** $f(x)g(x)$ without expanding

a $f(x) = 5x^2 + 4$ and $g(x) = x - 7$ **b** $f(x) = 3x^2 + 1$ and $g(x) = 2x - 5$

c $f(x) = (2x - 5)^2$ and $g(x) = 4x - 3$ **d** $f(x) = x^3 + 7$ and $g(x) = 2x^2$

4 Find the domain and range of $y = f(x) + g(x)$ given:

a $f(x) = x + 2$ and $g(x) = x - 4$ **b** $f(x) = 2x^2 + x - 1$ and $g(x) = -x - 1$

c $f(x) = x^3$ and $g(x) = x + 2$ **d** $f(x) = x^2 - 1$ and $g(x) = x - 1$

5 Find the domain and range of $y = f(x) - g(x)$ given:

a $f(x) = 3x + 2$ and $g(x) = x - 1$ **b** $f(x) = x^2 - 1$ and $g(x) = x - 1$

c $f(x) = x^3 + x$ and $g(x) = x + 2$ **d** $f(x) = 3x^2 - x - 1$ and $g(x) = x^2 + x + 3$

6 Find the domain and range of $y = f(x)g(x)$ given:

a $f(x) = x + 2$ and $g(x) = x - 4$ **b** $f(x) = x - 5$ and $g(x) = x + 5$

c $f(x) = x^2$ and $g(x) = x$

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7 Find the domain of $y = \dfrac{f(x)}{g(x)}$ given:

a $f(x) = 5$ and $g(x) = x - 4$

b $f(x) = x - 1$ and $g(x) = x + 1$

c $f(x) = 2x$ and $g(x) = x - 3$

d $f(x) = x + 3$ and $g(x) = x^3$

8 Find the composite function $f(g(x))$ given:

a $f(x) = x^2$ and $g(x) = x^2 + 1$

b $f(x) = x^3$ and $g(x) = 5x - 3$

c $f(x) = x^7$ and $g(x) = x^2 - 3x + 2$

d $f(x) = \sqrt{x}$ and $g(x) = 2x - 1$

e $f(x) = \sqrt[3]{x}$ and $g(x) = x^4 + 7x^2 - 4$

f $f(x) = 3x$ and $g(x) = 2x + 1$

g $f(x) = 2x - 7$ and $g(x) = x^3$

h $f(x) = 6x - 5$ and $g(x) = x^2$

i $f(x) = 2x^2$ and $g(x) = 3x$

j $f(x) = 4x^2 + 1$ and $g(x) = x^2 + 3$

9 Find the domain and range of the composite function $f(g(x))$ given that:

a $f(x) = x^2$ and $g(x) = x - 1$

b $f(x) = x^3$ and $g(x) = x + 5$

c $f(x) = \sqrt{x}$ and $g(x) = x - 2$

d $f(x) = -\sqrt{x}$ and $g(x) = 3x + 9$

e $f(x) = \sqrt{x}$ and $g(x) = 4 - x^2$

f $f(x) = -\sqrt{x}$ and $g(x) = 1 - x^2$

10 If $f(x) = \sqrt{x}$ and $g(x) = x^3$, find:

a $f(g(x))$

b $g(f(x))$

11 If $f(x) = \dfrac{1}{x}$ and $g(x) = x^2 + 3$, find:

a $y = f(x)g(x)$

b $y = f(g(x))$

c $y = \dfrac{f(x)}{g(x)}$

d $y = \dfrac{g(x)}{f(x)}$

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5. TEST YOURSELF

Practice quiz

For Questions **1** to **3**, select the correct answer **A**, **B**, **C** or **D**.

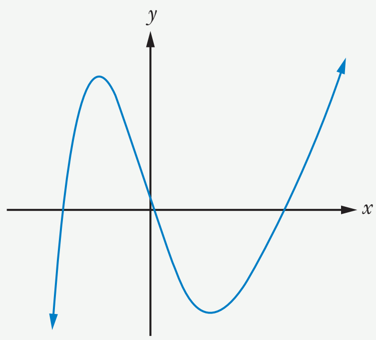
1 The domain of $y = -\dfrac{3}{x-4}$ is:

A (-4)
B $(-\infty, 4) \cup (4, \infty)$
C $(-\infty, -4) \cup (-4, \infty)$
D $(-\infty, 4)$

2 The equation of a circle with radius 3 and centre $(1, -2)$ is:

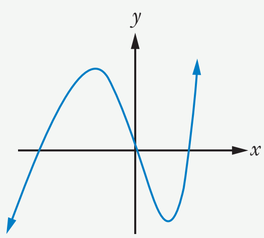
A $(x-1)^2 + (y+2)^2 = 9$
B $(x+1)^2 + (y-2)^2 = 9$
C $(x-1)^2 + (y+2)^2 = 3$
D $(x+1)^2 + (y-2)^2 = 3$

3 The graph of $y = f(x)$ is shown below.

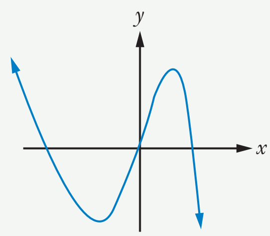


Which one of these is the graph of $y = -f(-x)$?

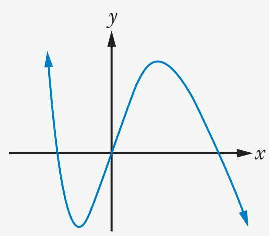
A



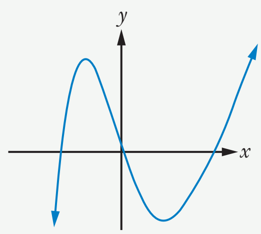
B



C



D



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4 The area of a pizza slice decreases as the number of people sharing it evenly increases. When 5 people share the pizza, the area of each slice is 30 cm^2.

a Find the equation of the area, A, of a pizza slice in terms of the number of people sharing, n.

b What is the area of one pizza slice when:

i 10 people share? **ii** 8 people share?

c How many people are sharing the pizza when each slice has an area of:

i 16. 67 cm^2? **ii** 25 cm^2?

5 Sketch the graph of each function or relation.

a $x^2 + y^2 = 1$ **b** $y = \frac{2}{x}$ **c** $y = |x + 2|$

d $y = -\sqrt{4 - x^2}$ **e** $y = f(-x)$ given $f(x) = \frac{2}{x - 1}$

f $y = -f(x)$ if $f(x) = 3x - 6$ **g** $y = -f(-x)$ given $f(x) = x^2 + x$

6 Find the radius and centre of the circle $x^2 - 6x + y^2 - 2y - 6 = 0$.

7 If $f(x) = x^3$ and $g(x) = 3x - 1$, find the equation of:

a $y = f(x) + g(x)$ **b** $y = f(x)g(x)$

c $y = f(g(x))$ **d** $y = g(f(x))$

8 **a** Is the circle $x^2 + y^2 = 1$ a function?

b Change the subject of the equation to y in terms of x.

c Sketch the graphs of 2 separate functions that together make up the circle $x^2 + y^2 = 1$.

9 Find the domain and range of each relation.

a $x^2 + y^2 = 16$ **b** $y = \frac{1}{x + 2}$

c $f(x) = |x| + 3$ **d** $y = \sqrt{9 - x^2}$

10 Find the domain and range of $y = f(x) + g(x)$ given $f(x) = x^2 - 4x$ and $g(x) = 2x - 3$.

11 **a** Write down the domain and range of the curve $y = \frac{2}{x - 3}$.

b Sketch the graph of $y = \frac{2}{x - 3}$.

12 **a** Sketch the graph of $y = |x + 1|$.

b From the graph, solve $|x + 1| = 3$

13 Solve graphically: $|x - 3| = 2$.

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14 Find the centre and radius of the circle with equation:

a $x^2 + y^2 = 100$

b $(x - 3)^2 + (y - 2)^2 = 121$

c $x^2 + 6x + y^2 + 2y + 1 = 0$

15 Find the x- and y-intercepts (where they exist) of:

a $P(x) = x^3 - 4x$

b $y = -\dfrac{2}{x+1}$

c $x^2 + y^2 = 9$

d $y = \sqrt{25 - x^2}$

e $f(x) = |x - 2| + 3$

16 If $f(x) = 2x^2 + x - 6$ and $g(x) = 5x^3 + 1$, find:

a the degree of $y = f(x) + g(x)$

b the leading term of $y = f(x)g(x)$

c the constant term of $y = f(x) - g(x)$

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5. CHALLENGE EXERCISE

1 Solve $|2x + 1| = 3x - 2$ graphically.

2 Given $f(x) = |x| + 3x - 4$, sketch the graph of:

a $y = f(x)$ **b** $y = -f(x)$

3 A variable a is inversely proportional to the square of b. When $b = 3$, $a = 2$.

a Find the equation of a in terms of b.

b Evaluate a when $b = 2$.

c Evaluate b when $a = 10$, correct to 2 decimal places, if $b > 0$.

4 Find the centre and radius of the circle with equation given by $x^2 + 3x + y^2 - 2y - 3 = 0$.

5 Find the equation of the straight line through the centres of the circles with equations $x^2 + 4x + y^2 - 8y - 5 = 0$ and $x^2 - 2x + y^2 + 10y + 10 = 0$.

6 Sketch the graph of $y = \dfrac{|x|}{x^2}$.

7 a Show that $\dfrac{2x+7}{x+3} = 2 + \dfrac{1}{x+3}$.

b Find the domain and range of $y = \dfrac{2x+7}{x+3}$.

c Hence sketch the graph of $y = \dfrac{2x+7}{x+3}$.

8 Show that $x^2 - 2x + y^2 + 4y + 1 = 0$ and $x^2 - 2x + y^2 + 4y - 4 = 0$ are concentric.

9 Sketch the graph of $f(x) = 1 - \dfrac{1}{x^2}$.

10 a Sketch the graph of $f(x) = \begin{cases} x & \text{for } x < -2 \\ x^2 & \text{for } -2 \le x \le 0 \\ 2 & \text{for } x > 0 \end{cases}$.

b Find any x values for which the function is discontinuous.

c Find the domain and range of the function.

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