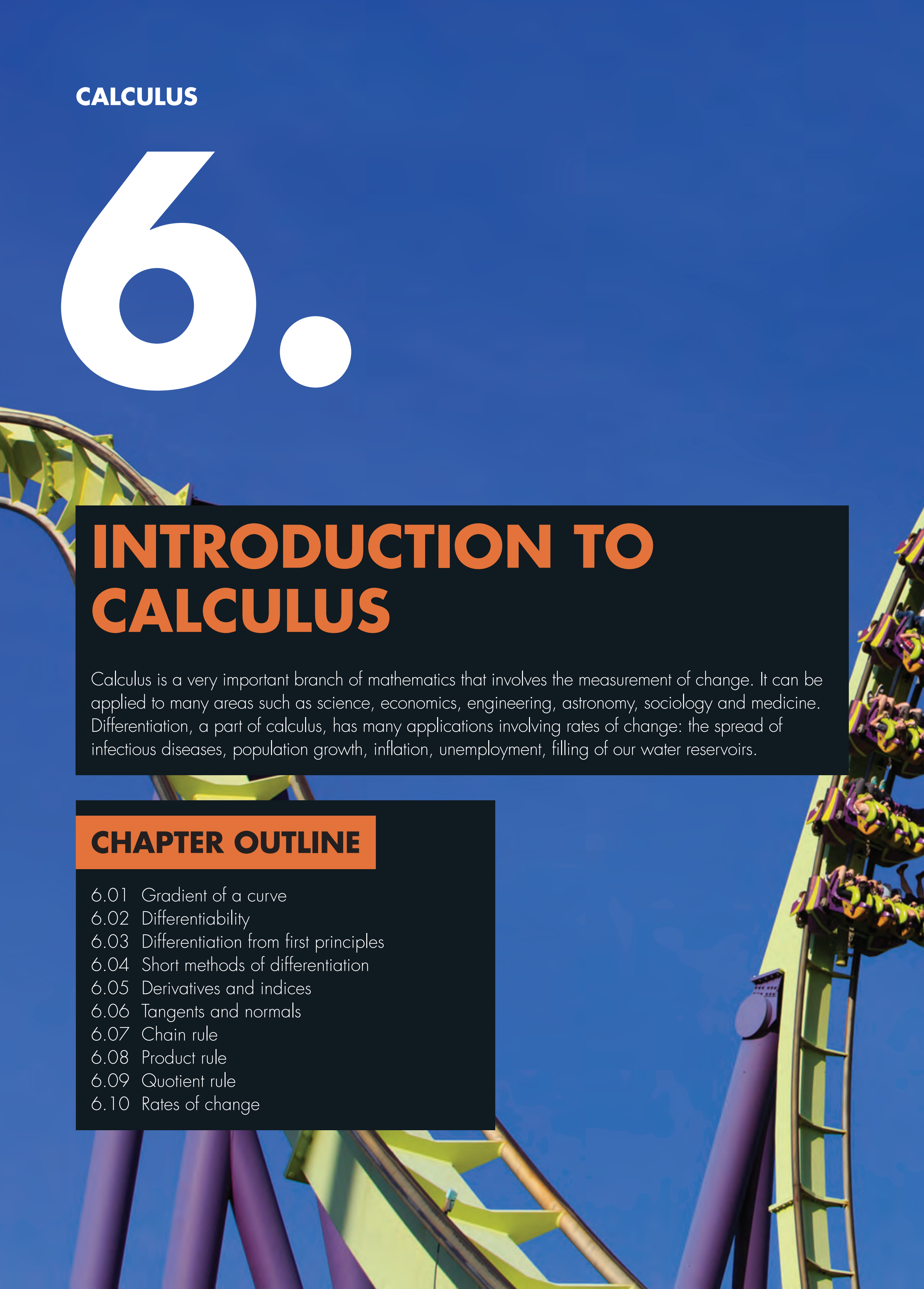
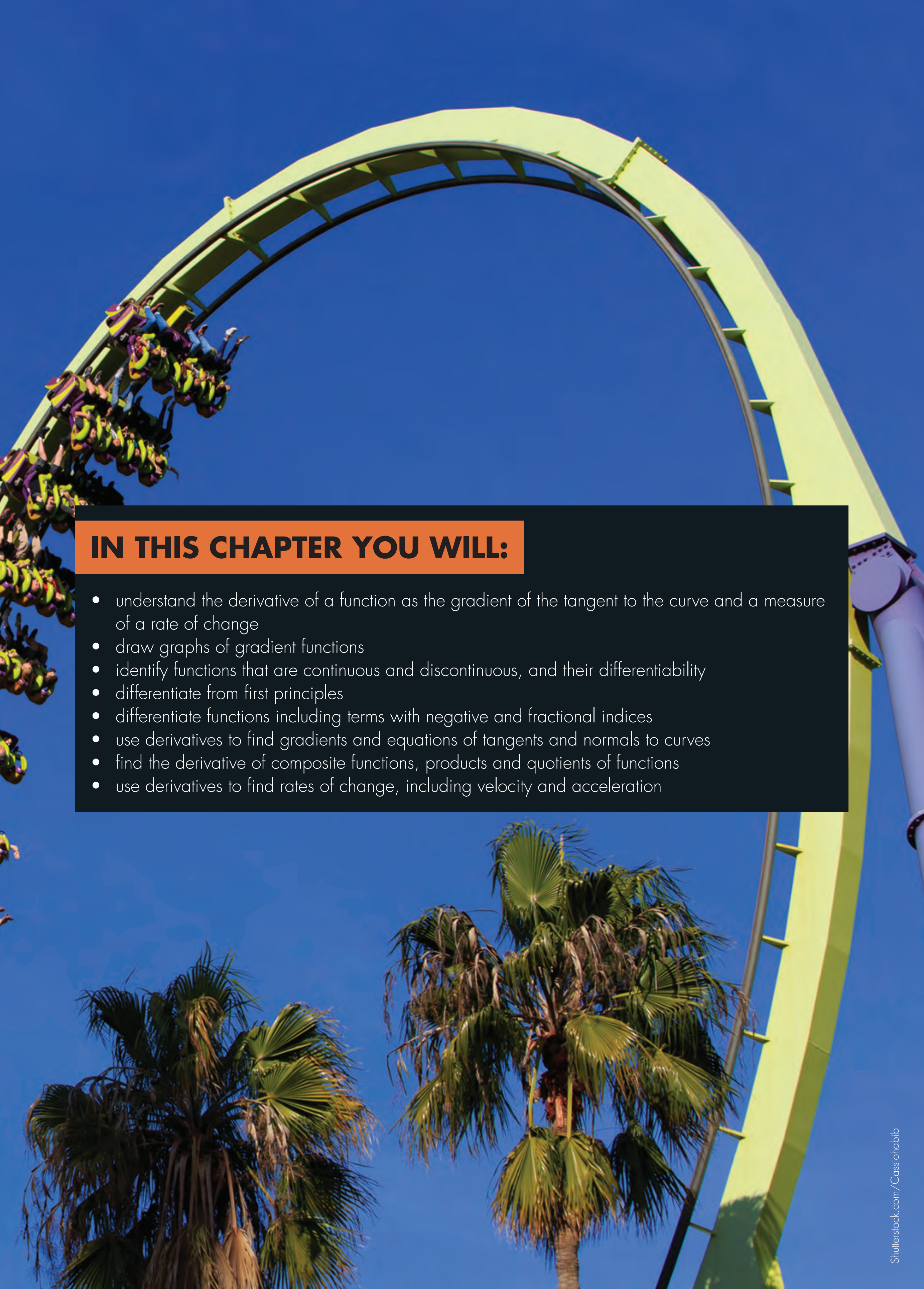


6. INTRODUCTION TO CALCULUS

Calculus is a very important branch of mathematics that involves the measurement of change. It can be applied to many areas such as science, economics, engineering, astronomy, sociology and medicine. Differentiation, a part of calculus, has many applications involving rates of change: the spread of infectious diseases, population growth, inflation, unemployment, filling of our water reservoirs.

CHAPTER OUTLINE



IN THIS CHAPTER YOU WILL:

- understand the derivative of a function as the gradient of the tangent to the curve and a measure of a rate of change
- draw graphs of gradient functions
- identify functions that are continuous and discontinuous, and their differentiability
- differentiate from first principles
- differentiate functions including terms with negative and fractional indices
- use derivatives to find gradients and equations of tangents and normals to curves
- find the derivative of composite functions, products and quotients of functions
- use derivatives to find rates of change, including velocity and acceleration

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TERMINOLOGY

acceleration: The rate of change of velocity with respect to time

average rate of change: The rate of change between 2 points on a function; the gradient of the line (secant) passing through those points

chain rule: A method for differentiating composite functions

derivative function: The gradient function $y = f'(x)$ of a function $y = f(x)$ obtained through differentiation

differentiability: A function is differentiable wherever its gradient is defined

differentiation: The process of finding the gradient function

differentiation from first principles: The process of finding the gradient of a tangent to a curve by finding the gradient of the secant between 2 points and finding the limit as the secant becomes a tangent

displacement: The distance and direction of an object in relation to the origin

gradient of a secant: The gradient (slope) of the line between 2 points on a function; measures the average rate of change between the 2 points

gradient of a tangent: The gradient of a line that is a tangent to the curve at a point on a function; measures the instantaneous rate of change of the function at that point

instantaneous rate of change: The rate of change at a particular point on a function; the gradient of the tangent at this point

limit: The value that a function approaches as the independent variable approaches some value

normal: A line that is perpendicular to the tangent at a given point on a curve

product rule: A method for differentiating the product of 2 functions

quotient rule: A method for differentiating the quotient of 2 functions

secant: A straight line passing through 2 points on the graph of a function

stationary point: A point on the graph of $y = f(x)$ where the tangent is horizontal and its gradient $f'(x) = 0$. It could be a maximum point, minimum point or a horizontal point of inflection

tangent: A straight line that just touches a curve at one point. The curve has the same gradient or direction as the tangent at that point

turning point: A maximum or minimum point on a curve, where the curve turns around

velocity: The rate of change of displacement of an object with respect to time; involves speed and direction

DID YOU KNOW?

Newton and Leibniz

‘Calculus’ comes from the Latin meaning ‘pebble’ or ‘small stone’. In many ancient civilisations stones were used for counting, but the mathematics they practised was quite sophisticated.

It was not until the 17th century that there was a breakthrough in calculus when scientists were searching for ways of measuring motion of objects such as planets, pendulums and projectiles.

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Isaac Newton

Isaac Newton (1642–1727), an Englishman, discovered the main principles of calculus when he was 23 years old. At this time an epidemic of bubonic plague had closed Cambridge University where he was studying, so many of his discoveries were made at home. He first wrote about his calculus methods, which he called fluxions, in 1671, but his *Method of fluxions* was not published until 1704.

Gottfried Leibniz (1646–1716), in Germany, was studying the same methods and there was intense rivalry between the two countries over who was first to discover calculus!

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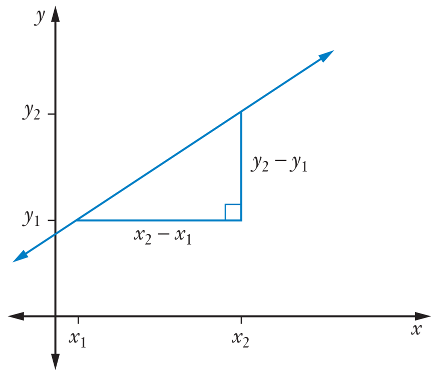
6.01 Gradient of a curve

WS Gradient functions

WS Gradient functions

The **gradient** of a straight line measures the **rate of change** of y (the dependent variable) with respect to the change in x (the independent variable).



Gradient

$$m = \frac{\text{rise}}{\text{run}}$$

$$= \frac{y_2 - y_1}{x_2 - x_1}$$

Notice that when the gradient of a straight line is positive the line is increasing, and when the gradient is negative the line is decreasing. Straight lines increase or decrease at a constant rate and the gradient is the same everywhere along the line.

CLASS DISCUSSION

Remember that an **increasing** line has a positive gradient and a **decreasing** line has a negative gradient.

What is the gradient of a horizontal line?

Can you find the gradient of a vertical line? Why?

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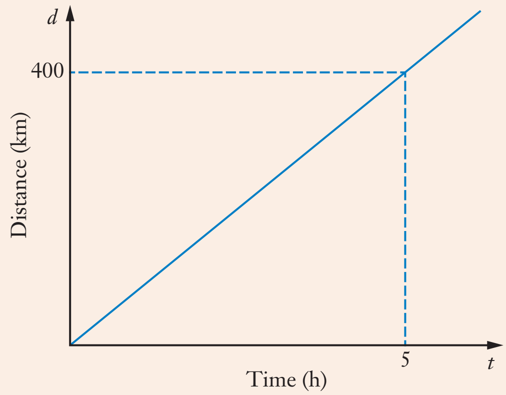
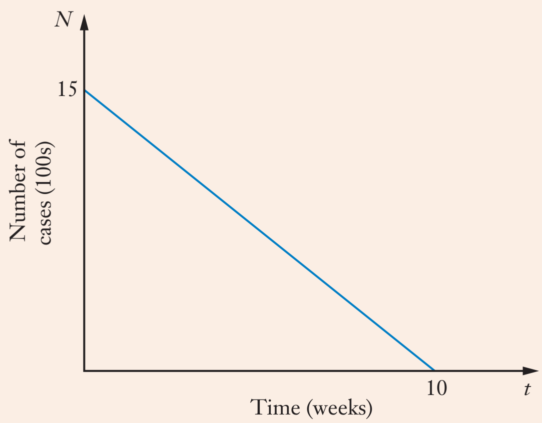
EXAMPLE 1

a The graph shows the distance travelled by a car over time. Find the gradient and describe it as a rate.



b The graph shows the number of cases of flu reported in a town over several weeks. Find the gradient and describe it as a rate.



Solution

a $m = \dfrac{\text{rise}}{\text{run}}$

$= \dfrac{400}{5}$

$= 80$

The line is increasing, so it has a positive gradient.

This means that the car is travelling at a constant rate (speed) of 80 km/h.

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b $m = \frac{\text{rise}}{\text{run}}$

$= \frac{-1500}{10}$

$= -150$

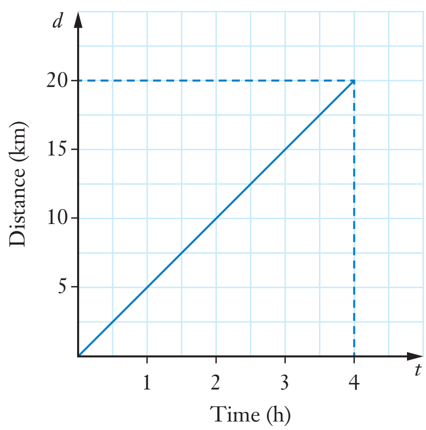
The line is decreasing, so it will have a negative gradient.

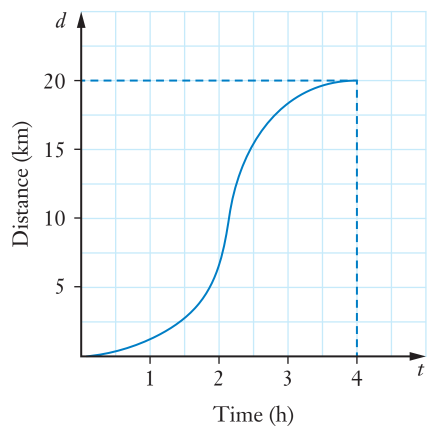
The 'rise' is a drop so it's negative.

This means that the rate is –150 cases/week, or the number of cases reported is decreasing by 150 cases/week.

CLASS DISCUSSION

The 2 graphs below show the distance that a bicycle travels over time. One is a straight line and the other is a curve.





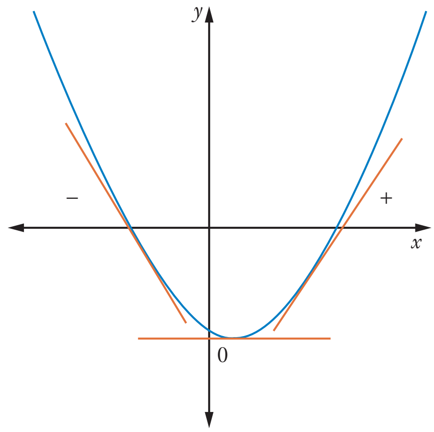
Is the average speed of the bicycle the same in both cases? What is different about the speed in the 2 graphs?

How could you measure the speed in the second graph at any one time? Does it change? If so, how does it change?

We can start finding rates of change along a curve by looking at its shape and how it behaves. We started looking at this in Chapter 3, *Functions*.

The gradient of a curve shows the **rate of change of y** as x changes. A **tangent** to a curve is a straight line that just touches the curve at one point. We can see where the gradient of a curve is positive, negative or zero by drawing **tangents to the curve** at different places around the curve and finding the gradients of the tangents.

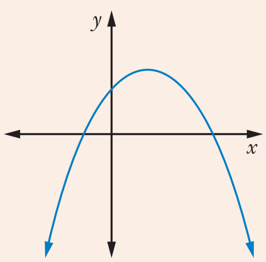
Notice that when the curve increases it has a positive gradient, when it decreases it has a negative gradient, and when it is a **turning point** the gradient is zero.



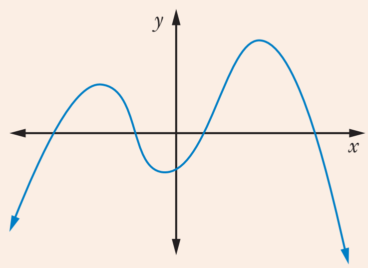
EXAMPLE 2

Copy each curve and write the sign of its gradient along the curve.

a



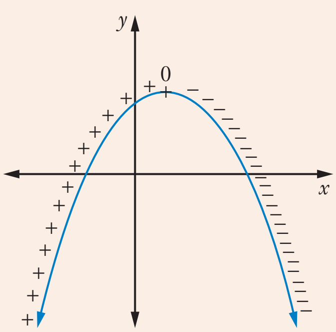
b



Solution

Where the curve increases, the gradient is positive. Where it decreases, it is negative. Where it is a turning point, it has a zero gradient.

a



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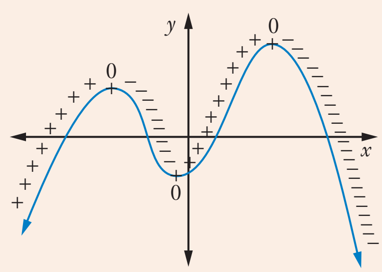

We find the gradient of a curve by measuring the **gradient of a tangent** to the curve at different points around the curve.

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We can then sketch the graph of these gradient values, which we call $y = f'(x)$, the **gradient function** or the **derivative function**.

EXAMPLE 3

a Make an accurate sketch of $f(x) = x^2$ on graph paper, or use graphing software.

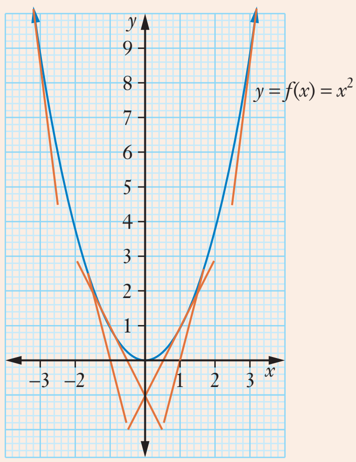
b Draw tangents to this curve at the points where $x = -3$, $x = -2$, $x = -1$, $x = 0$, $x = 1$, $x = 2$ and $x = 3$.

c Find the gradient of each of these tangents.

d Draw the graph of $y = f'(x)$ (the derivative or gradient function).

Solution

a and **b**



c At $x = -3$, $m = -6$

At $x = -2$, $m = -4$

At $x = -1$, $m = -2$

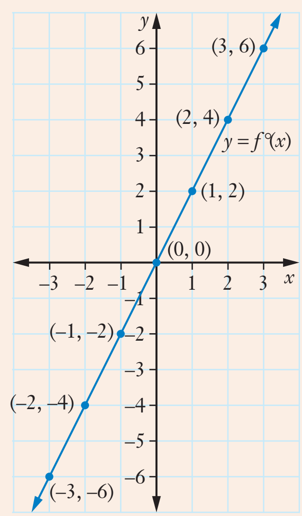
At $x = 0$, $m = 0$

At $x = 1$, $m = 2$

At $x = 2$, $m = 4$

At $x = 3$, $m = 6$

d Using the values from part **c**, $y = f'(x)$ is a linear function.



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Notice in Example 3 that where $m > 0$, the gradient function is above the x-axis; where $m = 0$, the gradient function is on the x-axis; and where $m < 0$, the gradient function is below the x-axis. Since $m = f'(x)$, we can write the following:

Sketching gradient (derivative) functions

$f'(x) > 0$: gradient function is above the x-axis

$f'(x) < 0$: gradient function is below the x-axis

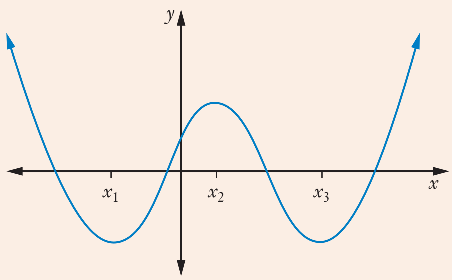
$f'(x) = 0$: gradient function is on the x-axis

Sketching gradient functions

EXAMPLE 4

Sketch a gradient function for each curve.

a



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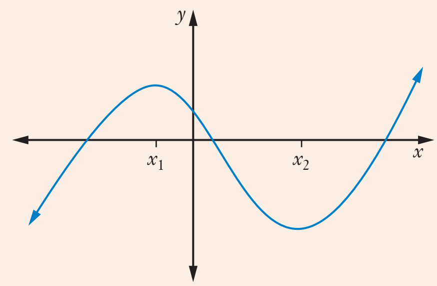
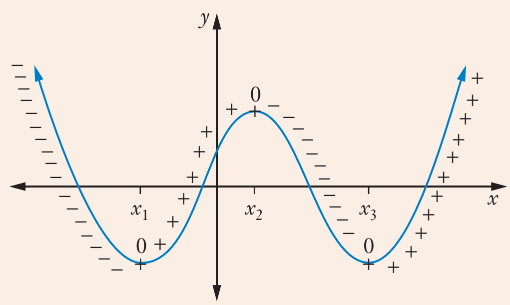
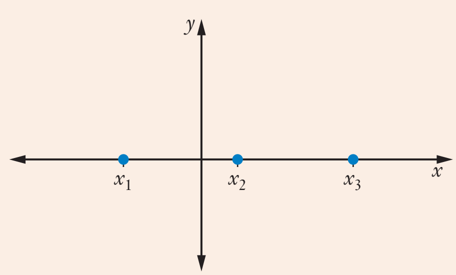


Solution

a First we mark in where the gradient is positive, negative and zero.



$f'(x) = 0$ at x_1, x_2 and x_3, so on the gradient graph these points will be on the x-axis (the x-intercepts of the gradient graph).



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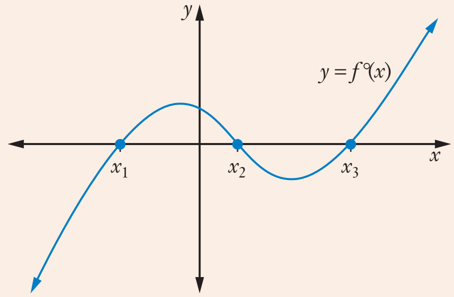
$f'(x) < 0$ to the left of x_1, so this part of the gradient graph will be below the x-axis.

$f'(x) > 0$ between x_1 and x_2, so the graph will be above the x-axis here.

$f'(x) < 0$ between x_2 and x_3, so the graph will be below the x-axis here.



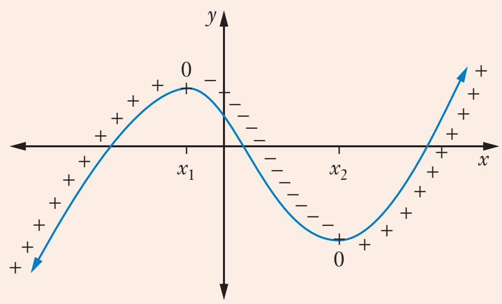
$f'(x) > 0$ to the right of x_3, so this part of the graph will be above the x-axis.

Sketching this information gives the graph of the gradient function $y = f'(x)$. Note that this is only a rough graph that shows the shape and sign rather than precise values.

b First mark in where the gradient is positive, negative and zero.

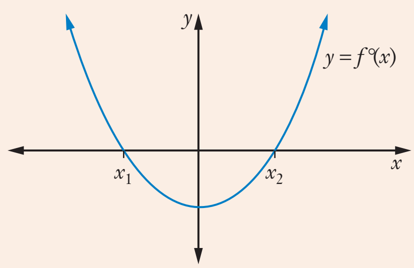
$f'(x) = 0$ at x_1 and x_2. These points will be the x-intercepts of the gradient function graph.

$f'(x) > 0$ to the left of x_1, so the graph will be above the x-axis here.



$f'(x) < 0$ between x_1 and x_2, so the graph will be below the x-axis here.

$f'(x) > 0$ to the right of x_2, so the graph will be above the x-axis here.



TECHNOLOGY

Tangents to a curve

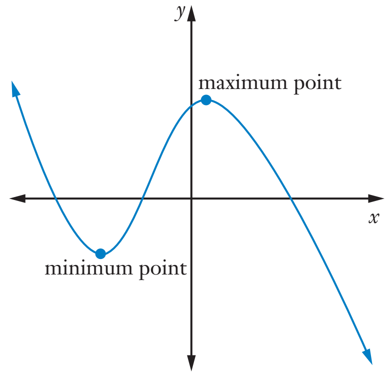
There are some excellent graphing software, online apps and websites that will draw tangents to a curve and sketch the gradient function.

Explore how to sketch gradient functions from the previous examples.

Stationary points

The points on a curve where the gradient $f'(x) = 0$ are called **stationary points** because the gradient there is neither increasing nor decreasing.

For example, the curve shown decreases to a **minimum turning point**, which is a type of **stationary point**. It then increases to a **maximum turning point** (also a stationary point) and then decreases again.



Exercise 6.01 Gradient of a curve

Sketch a gradient function for each curve.

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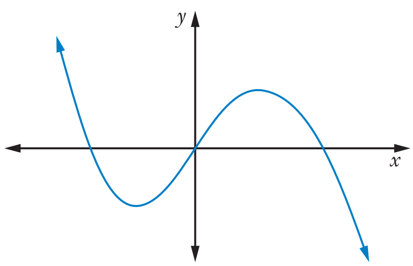

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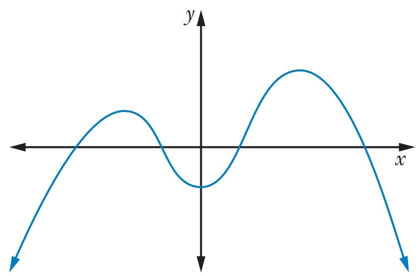

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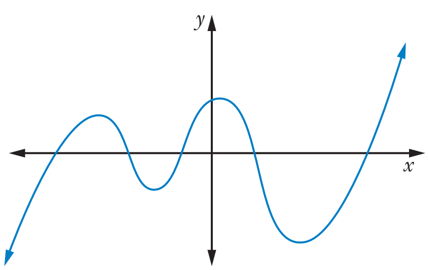

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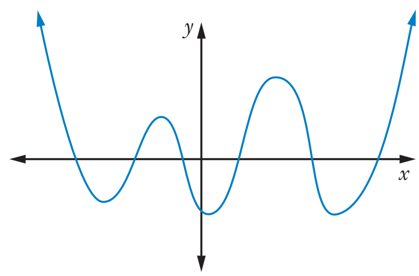

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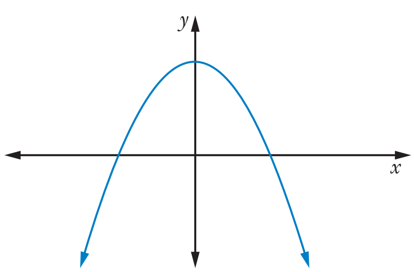

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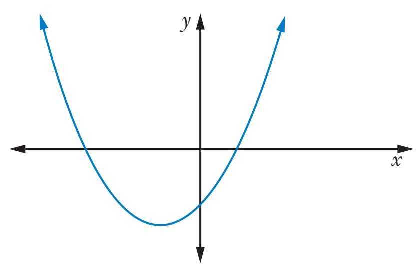

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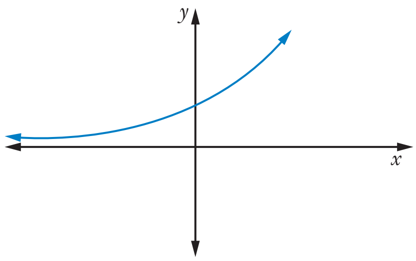

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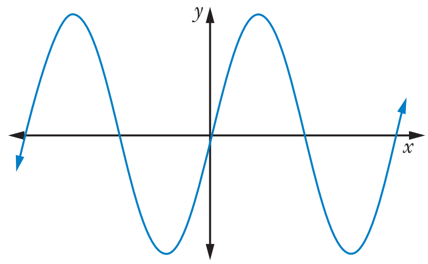

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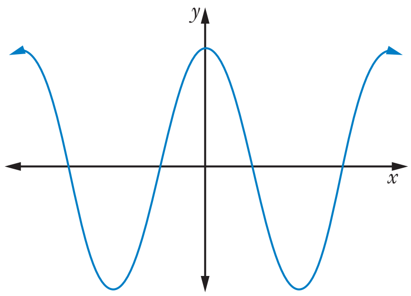

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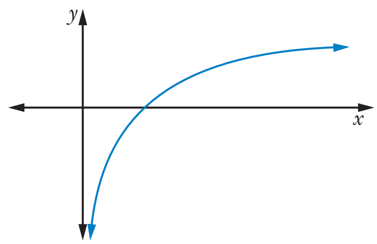


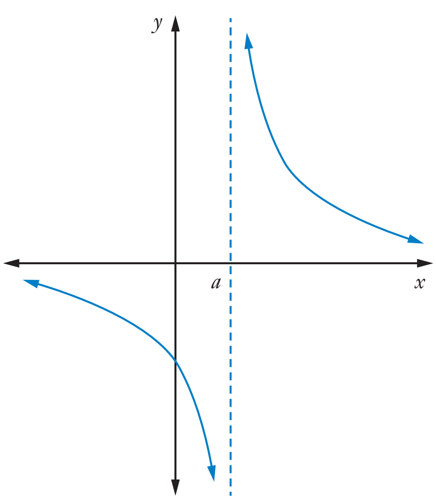
6.02 Differentiability

The process of finding the gradient function $y = f'(x)$ is called **differentiation**.

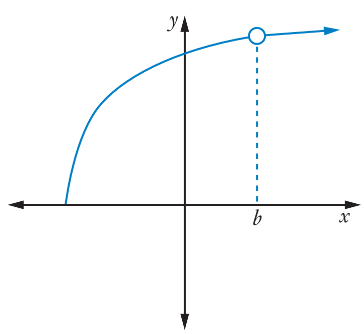
$y = f'(x)$ is called the **derivative function**, or just the **derivative**.

A function is **differentiable** at any point where it is continuous because we can find its gradient at that point. Linear, quadratic, cubic and other polynomial functions are differentiable at all points because their graphs are smooth and unbroken. A function is **not differentiable** at any point where it is **discontinuous**, where there is a gap or break in its graph.

This hyperbola is not differentiable at $x = a$ because the curve is discontinuous at this point.



This function is not differentiable at $x = b$ because the curve is discontinuous at this point.



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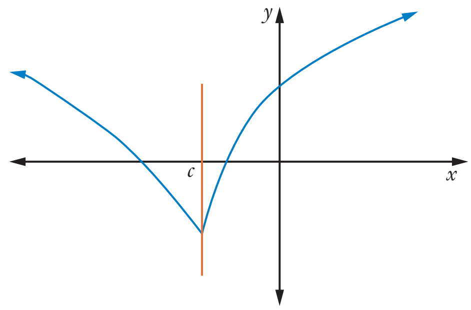
A function is also **not differentiable** where it is not smooth.

This function is not **differentiable** at $x = c$ since it is not smooth at that point. We cannot draw a unique tangent there so we cannot find the gradient of the function at that point.

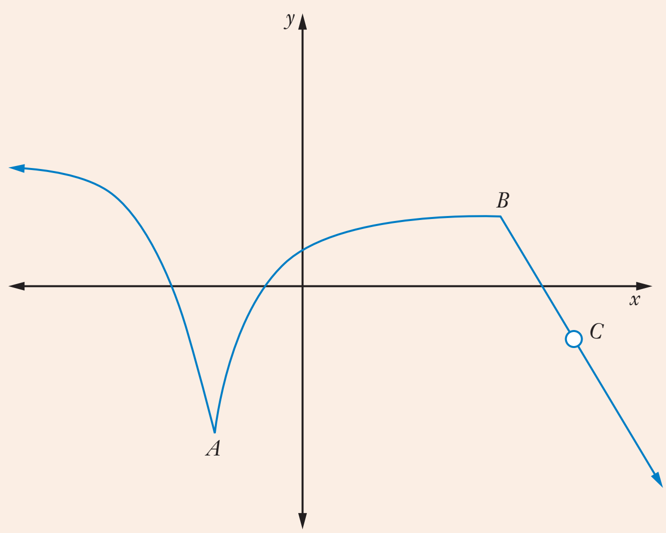


Differentiability at a point

A function $y = f(x)$ is **differentiable** at the point $x = a$ if its graph is **continuous** and **smooth** at $x = a$.

EXAMPLE 5

a Find all points where the function below is not differentiable.



b Is the function $f(x) = \begin{cases} x^2 & \text{for } x \geq 1 \\ 3x - 2 & \text{for } x < 1 \end{cases}$ differentiable at all points?

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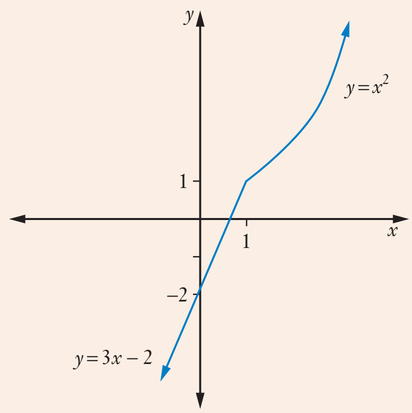
Solution

a The function is not differentiable at points A and B because the curve is not smooth at these points.

It is not differentiable at point C because the function is discontinuous at this point.

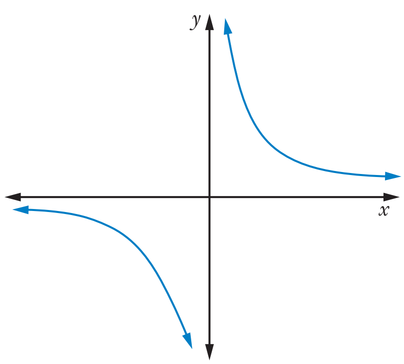
b Sketching this piecewise function shows that it is not smooth where the 2 parts meet, so it is not differentiable at $x = 1$.



Exercise 6.02 Differentiability

For each graph of a function, state any x values where the function is not differentiable.

1



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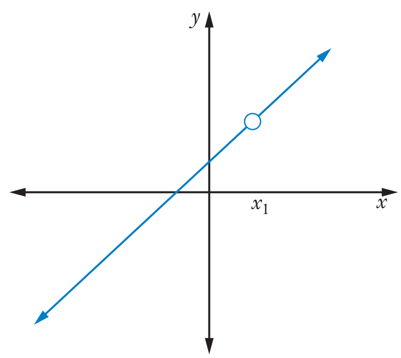

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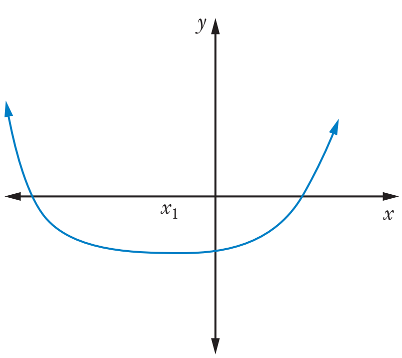

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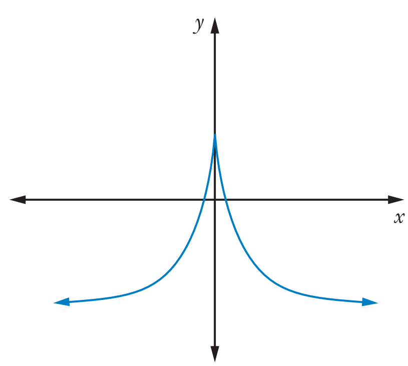

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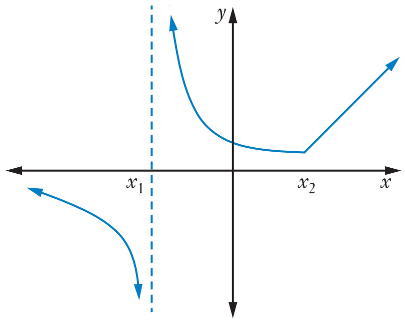


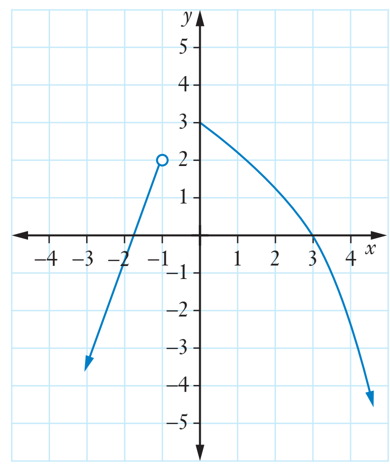
6 $f(x) = \frac{4}{x}$

7 $y = -\frac{1}{x+3}$

8 $f(x) = \begin{cases} x^3 & \text{for } x > 2 \\ x+1 & \text{for } x \le 2 \end{cases}$

9 $f(x) = \begin{cases} 2x & \text{for } x > 3 \\ 3 & \text{for } -2 \le x \le 3 \\ 1-x^2 & \text{for } x < -2 \end{cases}$

10



11 $f(x) = \frac{|x|}{x}$

WS Differentiation from first principles

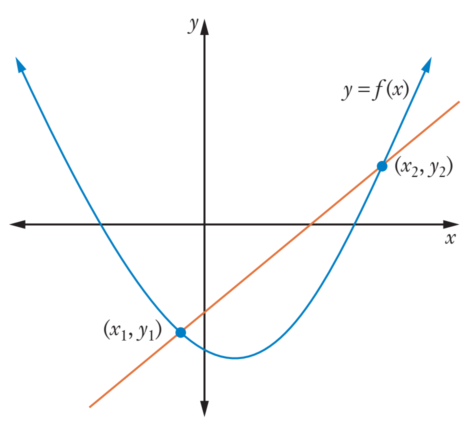
WS Limits

WS Finding derivatives from first principles

WS Rates of change - Gradients of secants

6.03 Differentiation from first principles

Gradient of a secant



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The line passing through the 2 points (x_1, y_1) and (x_2, y_2) on the graph of a function $y = f(x)$ is called a **secant**.

Differentiation from first principles

WS

Limits

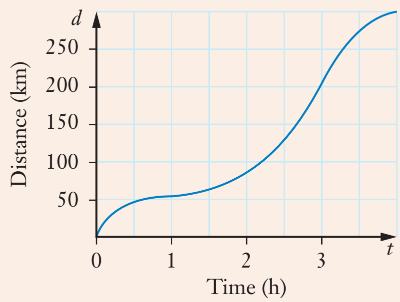
Gradient of the secant

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

The **gradient of a secant** gives the **average rate of change** between the 2 points.

EXAMPLE 6

a This graph shows the distance d in km that a car travels over time t in hours. After 1 hour the car has travelled 55 km and after 3 hours the car has travelled 205 km. Find the average speed of the car.



b Given the function $f(x) = x^2$, find the average rate of change between $x = 1$ and $x = 1.1$.

Solution

a Speed is the change in distance over time.

The gradient of the secant will give the average speed.

Average rate of change:

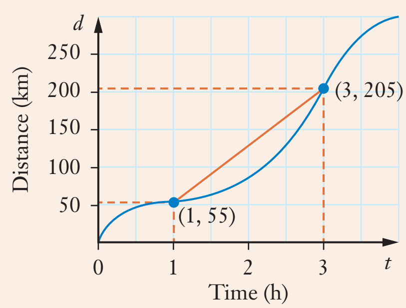
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{205 - 55}{3 - 1}$$

$$= \frac{150}{2}$$

$$= 75$$

So the average speed is 75 km/h.



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b When $x = 1$:

$f(1) = 1^2$

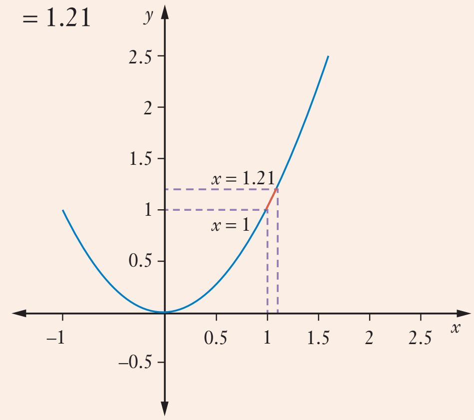
$= 1$

When $x = 1.1$:

$f(1.1) = 1.1^2$

$= 1.21$

So points are (1, 1) and (1.1, 1.21).



Average rate of change:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{1.21 - 1}{1.1 - 1}$$

$$= 2.1$$

So the average rate of change is 2.1.

Notice that the secant (orange interval) is very close to the shape of the curve itself. This is because the 2 points chosen are close together.

Estimating the gradient of a tangent

By taking 2 points close together, the **average rate of change** is quite close to the gradient of the tangent to the curve at one of those points, which is called the **instantaneous rate of change** at that point.

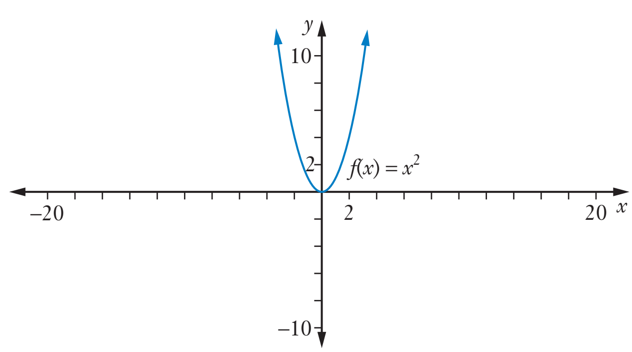
If you look at a close-up of a graph, you can get some idea of this concept. When the curve is magnified, any 2 points close together appear to be joined by a straight line. We say the curve is **locally straight**.

TECHNOLOGY

Locally straight curves

Use a graphics calculator or graphing software to sketch a curve and then zoom in on a section of the curve to see that it is locally straight.

For example, here is the parabola $y = x^2$.



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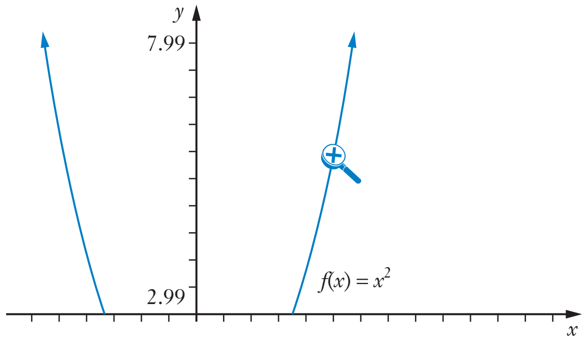
Notice how it looks straight when we zoom in on a point on the parabola.

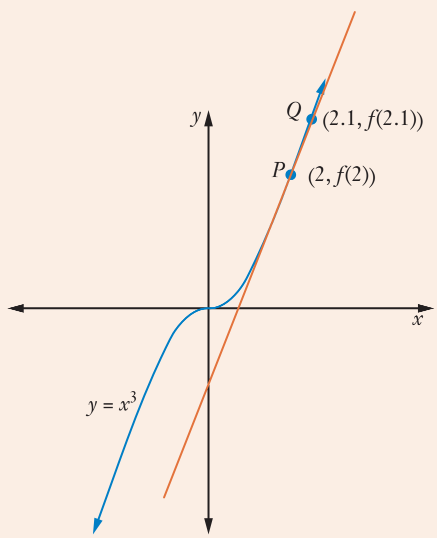


Use technology to sketch other curves and zoom in to show that they are locally straight.

We can calculate an approximate value for the gradient of the tangent at a point on a curve by taking another point close by, then calculating the gradient of the secant joining those 2 points.

EXAMPLE 7

a For $f(x) = x^3$, find the gradient of the secant PQ where P is the point on the curve where $x = 2$ and Q is another point on the curve where $x = 2.1$. Then choose different values for Q and use these results to estimate $f'(2)$, the gradient of the tangent to the curve at P.



b For the curve $y = x^2$, find the gradient of the secant AB where A is the point on the curve where $x = 5$ and point B is close to A. Find an estimate of the gradient of the tangent to the curve at A by using 3 different values for B.

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Solution

a P is $(2, f(2))$. Take different values of x for point Q, starting with $x = 2.1$, and find the gradient of the secant using $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.

Point Q	Gradient of secant PQ	Point Q	Gradient of secant PQ
$(2.1, f(2.1))$	$m = \dfrac{f(2.1) - f(2)}{2.1 - 2}$ $= \dfrac{2.1^3 - 2^3}{0.1}$ $= 12.61$	$(1.9, f(1.9))$	$m = \dfrac{f(1.9) - f(2)}{1.9 - 2}$ $= \dfrac{1.9^3 - 2^3}{-0.1}$ $= 11.41$
$(2.01, f(2.01))$	$m = \dfrac{f(2.01) - f(2)}{2.01 - 2}$ $= \dfrac{2.01^3 - 2^3}{0.01}$ $= 12.0601$	$(1.99, f(1.99))$	$m = \dfrac{f(1.99) - f(2)}{1.99 - 2}$ $= \dfrac{1.99^3 - 2^3}{-0.01}$ $= 11.9401$
$(2.001, f(2.001))$	$m = \dfrac{f(2.001) - f(2)}{2.001 - 2}$ $= \dfrac{2.001^3 - 2^3}{0.001}$ $= 12.006\,001$	$(1.999, f(1.999))$	$m = \dfrac{f(1.999) - f(2)}{1.999 - 2}$ $= \dfrac{1.999^3 - 2^3}{-0.001}$ $= 11.994\,001$

From these results, we can see that a good estimate for $f'(2)$, the gradient at P, is 12.

As $x \to 2, f'(2) \to 12$.

We use a special notation for **limits** to show this.

$$f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}$$
$$= 12$$

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b $A = (5, f(5))$

Take 3 different values of x for point B; for example, $x = 4.9$, $x = 5.1$ and $x = 5.01$.

$B = (4.9, f(4.9))$

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{f(4.9) - f(5)}{4.9 - 5} \\ &= \frac{4.9^2 - 5^2}{-0.1} \\ &= 9.9 \end{aligned}$$

$B = (5.1, f(5.1))$

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{f(5.1) - f(5)}{5.1 - 5} \\ &= \frac{5.1^2 - 5^2}{0.1} \\ &= 10.1 \end{aligned}$$

$B = (5.01, f(5.01))$

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{f(5.01) - f(5)}{5.01 - 5} \\ &= \frac{5.01^2 - 5^2}{0.01} \\ &= 10.01 \end{aligned}$$

As $x \to 5, f'(5) \to 10$.

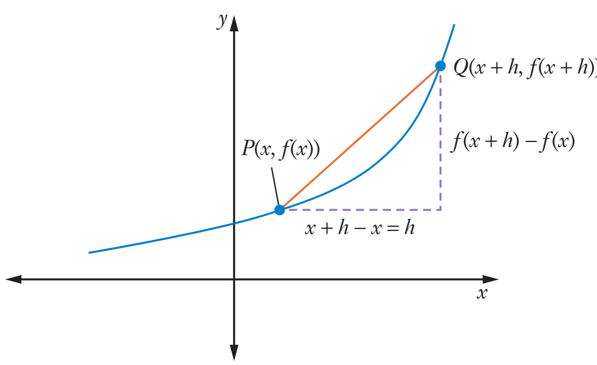
$$\begin{aligned} f'(5) &= \lim_{x \to 5} \frac{f(x) - f(5)}{x - 5} \\ &= 10 \end{aligned}$$

The difference quotient

We measure the instantaneous rate of change of any point on the graph of a function by using limits to find the gradient of the tangent to the curve at that point. This is called **differentiation from first principles**. Using the method from the examples above, we can find a general formula for the derivative function $y = f'(x)$.

We want to find the instantaneous rate of change or gradient of the tangent to a curve $y = f(x)$ at point $P(x, f(x))$.

We choose a second point Q close to P with coordinates $(x + h, f(x + h))$ where h is small.



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Now find the gradient of the secant PQ.

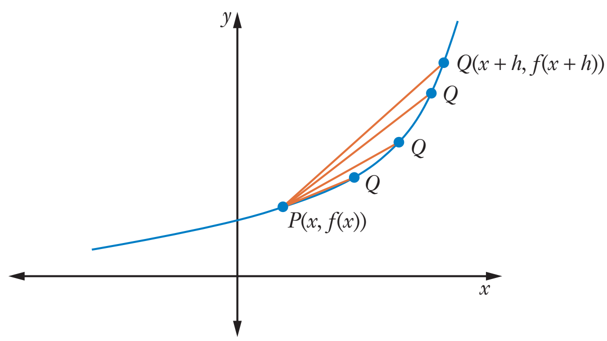
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{f(x+h) - f(x)}{x+h-x}$$
$$= \frac{f(x+h) - f(x)}{h}$$

$\frac{f(x+h) - f(x)}{h}$ is called the **difference quotient** and it gives an **average rate of change**.

To find the gradient of the tangent at P, we make h smaller as shown, so that Q becomes closer and closer to P.

As h approaches 0, the gradient of the tangent becomes $\lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

We call this $f'(x)$ or $\frac{dy}{dx}$ or y'.



Differentiation from first principles

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

INVESTIGATION

CALCULUS NOTATION

On p.274, we learned about the mathematicians Isaac Newton and Gottfried Leibniz. Newton used the notation $f'(x)$ for the derivative function while Leibniz used the notation $\frac{dy}{dx}$ where d stood for 'difference'. Can you see why he would have used this?

Use the Internet to explore the different notations used in calculus and where they came from.

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EXAMPLE 8

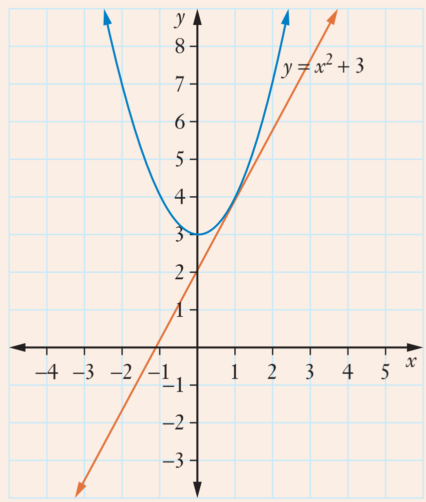
a Differentiate from first principles to find the gradient of the tangent to the curve $y = x^2 + 3$ at the point where $x = 1$.

b Differentiate $f(x) = 2x^2 + 7x - 3$ from first principles.

Solution

a

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$



$$f(x) = x^2 + 3$$

$$f(x+h) = (x+h)^2 + 3$$
$$= x^2 + 2xh + h^2 + 3$$

Substitute $x = 1$:

$$f(1) = 1^2 + 3$$
$$= 4$$

$$f(1+h) = 1^2 + 2(1)h + h^2 + 3$$
$$= 4 + 2h + h^2$$

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$
$$= \lim_{h \to 0} \frac{4 + 2h + h^2 - 4}{h}$$
$$= \lim_{h \to 0} \frac{2h + h^2}{h}$$
$$= \lim_{h \to 0} \frac{h(2+h)}{h}$$
$$= \lim_{h \to 0} (2+h)$$
$$= 2 + 0$$
$$= 2$$

So the gradient of the tangent to the curve $y = x^2 + 3$ at the point $(1, 4)$ is 2.

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b

$$f(x) = 2x^2 + 7x - 3$$
$$\begin{aligned}
f(x+h) &= 2(x+h)^2 + 7(x+h) - 3 \\
&= 2(x^2 + 2xh + h^2) + 7x + 7h - 3 \\
&= 2x^2 + 4xh + 2h^2 + 7x + 7h - 3 \\
f(x+h) - f(x) &= 2x^2 + 4xh + 2h^2 + 7x + 7h - 3 - (2x^2 + 7x - 3) \\
&= 2x^2 + 4xh + 2h^2 + 7x + 7h - 3 - 2x^2 - 7x + 3 \\
&= 4xh + 2h^2 + 7h \\
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{4xh + 2h^2 + 7h}{h} \\
&= \lim_{h \to 0} \frac{h(4x + 2h + 7)}{h} \\
&= \lim_{h \to 0} (4x + 2h + 7) \\
&= 4x + 0 + 7 \\
&= 4x + 7
\end{aligned}$$

So the gradient function (derivative) of $f(x) = 2x^2 + 7x - 3$ is $f'(x) = 4x + 7$.

Exercise 6.03 Differentiation from first principles

1 **a** For the curve $y = x^4 + 1$, find the gradient of the secant between the point (1, 2) and the point where $x = 1.01$.

b Find the gradient of the secant between (1, 2) and the point where $x = 0.999$ on the curve.

c Use these results to find an approximation to the gradient of the tangent to the curve $y = x^4 + 1$ at the point (1, 2).

2 For the function $f(x) = x^3 + x$, find the average rate of change between the point (2, 10) and the point on the curve where:

a $x = 2.1$ **b** $x = 2.01$ **c** $x = 1.99$

Hence find an approximation to the gradient of the tangent at the point (2, 10).

3 For the function $f(x) = x^2 - 4$, find the gradient of the tangent at point P where $x = 3$ by selecting points near P and finding the gradient of the secant.

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4 A function is given by $f(x) = x^2 + x + 5$.

a Find $f(2)$.

b Find $f(2 + h)$.

c Find $f(2 + h) - f(2)$.

d Show that $\dfrac{f(2+h)-f(2)}{h} = 5 + h$.

e Find $f'(2)$.

5 Given the curve $f(x) = 4x^2 - 3$, find:

a $f(-1)$

b $f(-1 + h) - f(-1)$

c the gradient of the tangent to the curve at the point where $x = -1$.

6 For the parabola $y = x^2 - 1$, find:

a $f(3)$

b $f(3 + h) - f(3)$

c $f'(3)$.

7 For the function $f(x) = 4 - 3x - 5x^2$, find:

a $f'(1)$

b the gradient of the tangent at the point $(-2, -10)$.

8 If $f(x) = x^2$:

a find $f(x + h)$

b show that $f(x + h) - f(x) = 2xh + h^2$

c show that $\dfrac{f(x+h)-f(x)}{h} = 2x + h$

d show that $f'(x) = 2x$.

9 A function is given by $f(x) = 2x^2 - 7x + 3$.

a Show that $f(x + h) = 2x^2 + 4xh + 2h^2 - 7x - 7h + 3$.

b Show that $f(x + h) - f(x) = 4xh + 2h^2 - 7h$.

c Show that $\dfrac{f(x+h)-f(x)}{h} = 4x + 2h - 7$.

d Find $f'(x)$.

10 Differentiate from first principles to find the gradient of the tangent to the curve:

a $f(x) = x^2$ at the point where $x = 1$

b $y = x^2 + x$ at the point $(2, 6)$

c $f(x) = 2x^2 - 5$ at the point where $x = -3$

d $y = 3x^2 + 3x + 1$ at the point where $x = 2$

e $f(x) = x^2 - 7x - 4$ at the point $(-1, 4)$.

11 Find the derivative function for each function from first principles.

a $f(x) = x^2$

b $y = x^2 + 5x$

c $f(x) = 4x^2 - 4x - 3$

d $y = 5x^2 - x - 1$

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6.04 Short methods of differentiation

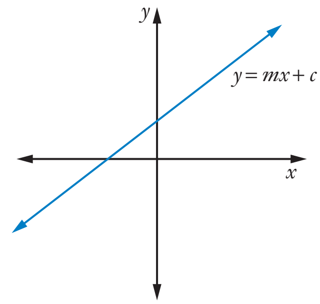
WS Derivatives of linear products

WS Derivatives of polynomials

Derivative of x^n

Remember that the gradient of a straight line $y = mx + c$ is m. The tangent to the line is the line itself, so the gradient of the tangent is m everywhere along the line.

So if $y = mx$, $\frac{dy}{dx} = m$.

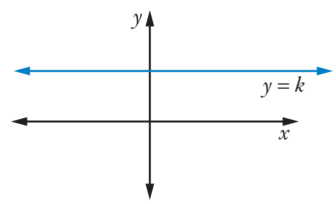


Derivative of *kx*

$$\frac{d}{dx}(kx) = k$$

A horizontal line $y = k$ has a gradient of zero.

So if $y = k$, $\frac{dy}{dx} = 0$.



Derivative of *k*

$$\frac{d}{dx}(k) = 0$$

TECHNOLOGY

Differentiation of powers of *X*

Find an approximation to the derivative of power functions such as $y = x^2$, $y = x^3$, $y = x^4$, $y = x^5$ by drawing the graph of $y = \frac{f(x+0.01) - f(x)}{0.01}$. You could use a graphics calculator or graphing software/website to sketch the derivative for these functions and find its equation. Can you find a pattern? Could you predict what the result would be for x^n?

When differentiating $y = x^n$ from first principles, a simple pattern appears:

- For $y = x$, $\quad f'(x) = 1x^0 = 1$
- For $y = x^2$, $\quad f'(x) = 2x^1 = 2x$
- For $y = x^3$, $\quad f'(x) = 3x^2$
- For $y = x^4$, $\quad f'(x) = 4x^3$
- For $y = x^5$, $\quad f'(x) = 5x^4$

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Derivative of x^n

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

If $y = x^n$ then $\dfrac{dy}{dx} = nx^{n-1}$.

There are some more properties of differentiation.

Derivative of kx^n

$$\frac{d}{dx}(kx^n) = knx^{n-1}$$

More generally:

Derivative of a constant multiple of a function

$$\frac{d}{dx}\left(kf(x)\right) = kf'(x)$$

EXAMPLE 9

a Find the derivative of $3x^8$.

b Differentiate $f(x) = 7x^3$.

Solution

a
$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$
$$\frac{d}{dx}(3x^8) = 3\times 8x^{8-1}$$
$$= 24x^7$$

b
$$f'(x) = knx^{n-1}$$
$$f'(x) = 7\times 3x^{3-1}$$
$$= 21x^2$$

If there are several terms in an expression, we differentiate each one separately.

Derivative of a sum of functions

$$\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$$

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EXAMPLE 10

a Differentiate $x^3 + x^4$.

b Find the derivative of $7x$.

c Differentiate $f(x) = x^4 - x^3 + 5$.

d Find the derivative of $y = 4x^7$.

e If $f(x) = 2x^5 - 7x^3 + 5x - 4$, evaluate $f'(-1)$.

f Find the derivative of $f(x) = 2x^2(3x - 7)$.

g Find the derivative of $\dfrac{3x^2 + 5x}{2x}$.

h Differentiate $S = 6r^2 - 12r$ with respect to r.

Solution

a $\dfrac{d}{dx}(x^3 + x^4) = 3x^2 + 4x^3$

b $\dfrac{d}{dx}(7x) = 7$

c
$$\begin{aligned} f'(x) &= 4x^3 - 3x^2 + 0 \\ &= 4x^3 - 3x^2 \end{aligned}$$

d
$$\begin{aligned} \frac{dy}{dx} &= 4 \times 7x^6 \\ &= 28x^6 \end{aligned}$$

e
$$\begin{aligned} f'(x) &= 10x^4 - 21x^2 + 5 \\ f'(-1) &= 10(-1)^4 - 21(-1)^2 + 5 \\ &= -6 \end{aligned}$$

f Expand first.
$$\begin{aligned} f(x) &= 2x^2(3x - 7) \\ &= 6x^3 - 14x^2 \\ f'(x) &= 18x^2 - 28x \end{aligned}$$

g Simplify first.
$$\begin{aligned} \frac{3x^2 + 5x}{2x} &= \frac{3x^2}{2x} + \frac{5x}{2x} \\ &= \frac{3x}{2} + \frac{5}{2} \\ \frac{d}{dx}\left(\frac{3x^2 + 5x}{2x}\right) &= \frac{3}{2} \\ &= 1\frac{1}{2} \end{aligned}$$

h Differentiating with respect to r rather than x:
$$\begin{aligned} S &= 6r^2 - 12r \\ \frac{dS}{dr} &= 12r - 12 \end{aligned}$$

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INVESTIGATION

FAMILIES OF CURVES

1 Differentiate:

a $x^2 + 1$ **b** $x^2 - 3$ **c** $x^2 + 7$
d x^2 **e** $x^2 + 20$ **f** $x^2 - 100$

What do you notice?

2 Differentiate:

a $x^3 + 5$ **b** $x^3 + 11$ **c** $x^3 - 1$
d $x^3 - 6$ **e** x^3 **f** $x^3 + 15$

What do you notice?

These groups of functions are families because they have the same derivatives.
Can you find others?

Exercise 6.04 Short methods of differentiation

1 Differentiate:

a $x + 2$ **b** $5x - 9$ **c** $x^2 + 3x + 4$
d $5x^2 - x - 8$ **e** $x^3 + 2x^2 - 7x - 3$ **f** $2x^3 - 7x^2 + 7x - 1$
g $3x^4 - 2x^2 + 5x$ **h** $x^6 - 5x^5 - 2x^4$ **i** $2x^5 - 4x^3 + x^2 - 2x + 4$
j $4x^{10} - 7x^9$

2 Find the derivative of:

a $x(2x + 1)$ **b** $(2x - 3)^2$ **c** $(x + 4)(x - 4)$
d $(2x^2 - 3)^2$ **e** $(2x + 5)(x^2 - x + 1)$

3 Find the derivative of:

a $\dfrac{x^2}{6} - x$ **b** $\dfrac{x^4}{2} - \dfrac{x^3}{3} + 4$ **c** $\dfrac{1}{3}x^6(x^2 - 3)$

d $\dfrac{2x^3 + 5x}{x}$ **e** $\dfrac{x^2 + 2x}{4x}$ **f** $\dfrac{2x^5 - 3x^4 + 6x^3 - 2x^2}{3x^2}$

4 Find $f'(x)$ when $f(x) = 8x^2 - 7x + 4$.

5 If $y = x^4 - 2x^3 + 5$, find $\dfrac{dy}{dx}$ when $x = -2$.

6 Find $\dfrac{dy}{dx}$ if $y = 6x^{10} - 5x^8 + 7x^5 - 3x + 8$.

7 If $s = 5t^2 - 20t$, find $\dfrac{ds}{dt}$.

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8 Find $g'(x)$ given $g(x) = 5x^4$.

9 Find $\frac{dv}{dt}$ when $v = 15t^2 - 9$.

10 If $h = 40t - 2t^2$, find $\frac{dh}{dt}$.

11 Given $V = \frac{4}{3}\pi r^3$, find $\frac{dV}{dr}$.

12 If $f(x) = 2x^3 - 3x + 4$, evaluate $f'(1)$.

13 Given $f(x) = x^2 - x + 5$, evaluate:

a $f'(3)$ **b** $f'(-2)$ **c** x when $f'(x) = 7$

14 If $y = x^3 - 7$, evaluate:

a the derivative when $x = 2$ **b** x when $\frac{dy}{dx} = 12$

15 Evaluate $g'(2)$ when $g(t) = 3t^3 - 4t^2 - 2t + 1$.

DID YOU KNOW?

Motion and calculus

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Galileo

Galileo (1564–1642) was very interested in the behaviour of bodies in motion. He dropped stones from the Leaning Tower of Pisa to try to prove that they would fall with equal speed. He rolled balls down slopes to prove that they move with uniform speed until friction slows them down. He showed that a body moving through the air follows a curved path at a fairly constant speed.

John Wallis (1616–1703) continued this study with his publication *Mechanica, sive Tractatus de Motu Geometricus*. He applied mathematical principles to the laws of motion and stimulated interest in the subject of mechanics.

Soon after Wallis' publication, **Christiaan Huygens** (1629–1695) wrote *Horologium Oscillatorium sive de Motu Pendulorum*, in which he described various mechanical principles. He invented the pendulum clock, improved the telescope and investigated circular motion and the descent of heavy bodies.

These three mathematicians provided the foundations of mechanics. **Sir Isaac Newton** (1642–1727) used calculus to increase the understanding of the laws of motion. He also used these concepts as a basis for his theories on gravity and inertia.

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6.05 Derivatives and indices

INVESTIGATION

DERIVATIVES AND INDICES

1 **a** Show that $\frac{1}{x+h}-\frac{1}{x}=\frac{-h}{x(x+h)}$.

b Hence differentiate $y=\frac{1}{x}$ from first principles.

c Differentiate $y=x^{-1}$ using the formula. Do you get the same answer as in part **b**?

2 **a** Show that $\left(\sqrt{x+h}-\sqrt{x}\right)\left(\sqrt{x+h}+\sqrt{x}\right)=h$.

b Hence differentiate $y=\sqrt{x}$ from first principles.

c Differentiate $y=x^{\frac{1}{2}}$ and show that this gives the same answer as in part **b**.

EXAMPLE 11

a Differentiate $f(x)=7\sqrt[3]{x}$.

b Find the derivative of $y=\frac{4}{x^2}$ at the point where $x=2$.

Solution

a

$$f(x)=7\sqrt[3]{x}=7x^{\frac{1}{3}}$$

Convert the function to a power of x first.

$$\begin{aligned}
f'(x)&=7\times\frac{1}{3}x^{\frac{1}{3}-1}\\
&=\frac{7}{3}x^{-\frac{2}{3}}\\
&=\frac{7}{3}\times\frac{1}{x^{\frac{2}{3}}}\\
&=\frac{7}{3}\times\frac{1}{\sqrt[3]{x^2}}\\
&=\frac{7}{3\sqrt[3]{x^2}}
\end{aligned}$$

b

$$\begin{aligned}
y&=\frac{4}{x^2}\\
&=4x^{-2}\\
\frac{dy}{dx}&=-8x^{-3}\\
&=-\frac{8}{x^3}
\end{aligned}$$

When $x=2$:

$$\begin{aligned}
\frac{dy}{dx}&=-\frac{8}{2^3}\\
&=-1
\end{aligned}$$

Exercise 6.05 Derivatives and indices

1 Differentiate:

a x^{-3} **b** $x^{1.4}$ **c** $6x^{0.2}$ **d** $x^{\frac{1}{2}}$

e $2x^{\frac{1}{2}} - 3x^{-1}$ **f** $3x^{\frac{1}{3}}$ **g** $8x^{\frac{3}{4}}$ **h** $-2x^{-\frac{1}{2}}$

2 Find the derivative function.

a $\frac{1}{x}$ **b** $5\sqrt{x}$ **c** $\sqrt[6]{x}$ **d** $\frac{2}{x^5}$

e $-\frac{5}{x^3}$ **f** $\frac{1}{\sqrt{x}}$ **g** $\frac{1}{2x^6}$ **h** $x\sqrt{x}$

i $\frac{2}{3x}$ **j** $\frac{1}{4x^2} + \frac{3}{x^4}$

3 Find the derivative of $y = \sqrt[3]{x}$ at the point where $x = 27$.

4 If $x = \frac{12}{t}$, find $\frac{dx}{dt}$ when $t = 2$.

5 A function is given by $f(x) = \sqrt[4]{x}$. Evaluate $f'(16)$.

6 Find the derivative of $y = \frac{3}{2x^2}$ at the point $\left(1, 1\frac{1}{2}\right)$.

7 Find $\frac{dy}{dx}$ if $y = \left(x + \sqrt{x}\right)^2$.

8 A function $f(x) = \frac{\sqrt{x}}{2}$ has a tangent at (4, 1). Find its gradient.

9 **a** Differentiate $\frac{\sqrt{x}}{x}$.

b Hence find the derivative of $y = \frac{\sqrt{x}}{x}$ at the point where $x = 4$.

10 The function $f(x) = 3\sqrt{x}$ has $f'(x) = \frac{3}{4}$ at $x = a$. Find a.

11 The hyperbola $y = \frac{2}{x}$ has 2 tangents with gradient $-\frac{2}{25}$. Find the points where these tangents touch the hyperbola.

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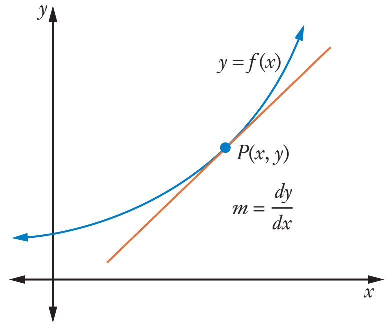
6.06 Tangents and normals

Tangents to a curve

Remember that the derivative is a function that gives the instantaneous rate of change or gradient of the tangent to the curve.

A tangent is a line so we can use the formula $y = mx + c$ or $y - y_1 = m(x - x_1)$ to find its equation.



Tangents and normals

Equation of a tangent

Slopes of curves

Tangents to a curve

EXAMPLE 12

a Find the gradient of the tangent to the parabola $y = x^2 + 1$ at the point (1, 2).

b Find values of x for which the gradient of the tangent to the curve $y = 2x^3 - 6x^2 + 1$ is equal to 18.

c Find the equation of the tangent to the curve $y = x^4 - 3x^3 + 7x - 2$ at the point (2, 4).

Solution

a The gradient of a tangent to a curve is $\frac{dy}{dx}$.

$$\frac{dy}{dx} = 2x + 0$$
$$= 2x$$

Substitute $x = 1$ from the point (1, 2):

$$\frac{dy}{dx} = 2(1)$$
$$= 2$$

So the gradient of the tangent at (1, 2) is 2.

b $\frac{dy}{dx} = 6x^2 - 12x$

Gradient is 18 so $\frac{dy}{dx} = 18$.

$$18 = 6x^2 - 12x$$
$$0 = 6x^2 - 12x - 18$$
$$x^2 - 2x - 3 = 0$$
$$(x - 3)(x + 1) = 0$$
$$\therefore x = 3, -1$$

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c $\dfrac{dy}{dx} = 4x^3 - 9x^2 + 7$

At (2, 4), $\dfrac{dy}{dx} = 4(2)^3 - 9(2)^2 + 7$

$= 3$

So the gradient of the tangent at (2, 4) is 3.

Equation of the tangent:

$y - y_1 = m(x - x_1)$

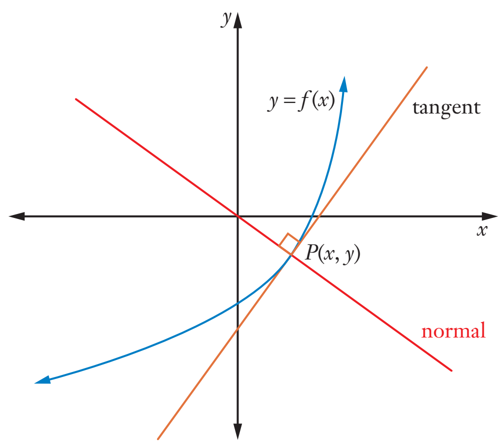
$y - 4 = 3(x - 2)$

$= 3x - 6$

$y = 3x - 2$ or $3x - y - 2 = 0$

Normals to a curve

The **normal** is a straight line **perpendicular** to the tangent at the same point of contact with the curve.



Remember the rule for perpendicular lines from Chapter 3, *Functions*:

Gradients of perpendicular lines

If 2 lines with gradients m_1 and m_2 are perpendicular, then $m_1 m_2 = -1$ or $m_2 = -\dfrac{1}{m_1}$.

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EXAMPLE 13

a Find the gradient of the normal to the curve $y = 2x^2 - 3x + 5$ at the point where $x = 4$.

b Find the equation of the normal to the curve $y = x^3 + 3x^2 - 2x - 1$ at $(-1, 3)$.

Solution

a $\frac{dy}{dx} = 4x - 3$

When $x = 4$:

$$\begin{aligned}\frac{dy}{dx} &= 4 \times 4 - 3 \\ &= 13\end{aligned}$$

So $m_1 = 13$

The normal is perpendicular to the tangent, so $m_1 m_2 = -1$.

$$\begin{aligned}13m_2 &= -1 \\ m_2 &= -\frac{1}{13}\end{aligned}$$

So the gradient of the normal is $-\frac{1}{13}$.

b $\frac{dy}{dx} = 3x^2 + 6x - 2$

When $x = -1$:

$$\begin{aligned}\frac{dy}{dx} &= 3(-1)^2 + 6(-1) - 2 \\ &= -5\end{aligned}$$

So $m_1 = -5$

The normal is perpendicular to the tangent, so $m_1 m_2 = -1$.

$$\begin{aligned}-5m_2 &= -1 \\ m_2 &= \frac{1}{5}\end{aligned}$$

So the gradient of the normal is $\frac{1}{5}$.

Equation of the normal: $y - y_1 = m(x - x_1)$

$$\begin{aligned}y - 3 &= \frac{1}{5}(x - (-1)) \\ 5y - 15 &= x + 1 \\ x - 5y + 16 &= 0\end{aligned}$$

Exercise 6.06 Tangents and normals

1 Find the gradient of the tangent to the curve:

a $y = x^3 - 3x$ at the point where $x = 5$

b $f(x) = x^2 + x - 4$ at the point $(-7, 38)$

c $f(x) = 5x^3 - 4x - 1$ at the point where $x = -1$

d $y = 5x^2 + 2x + 3$ at $(-2, 19)$

e $y = 2x^9$ at the point where $x = 1$

f $f(x) = x^3 - 7$ at the point where $x = 3$

g $v = 2t^2 + 3t - 5$ at the point where $t = 2$

h $Q = 3r^3 - 2r^2 + 8r - 4$ at the point where $r = 4$

i $h = t^4 - 4t$ where $t = 0$

j $f(t) = 3t^5 - 8t^3 + 5t$ at the point where $t = 2$.

2 Find the gradient of the normal to the curve:

a $f(x) = 2x^3 + 2x - 1$ at the point where $x = -2$

b $y = 3x^2 + 5x - 2$ at $(-5, 48)$

c $f(x) = x^2 - 2x - 7$ at the point where $x = -9$

d $y = x^3 + x^2 + 3x - 2$ at $(-4, -62)$

e $f(x) = x^{10}$ at the point where $x = -1$

f $y = x^2 + 7x - 5$ at $(-7, -5)$

g $A = 2x^3 + 3x^2 - x + 1$ at the point where $x = 3$

h $f(a) = 3a^2 - 2a - 6$ at the point where $a = -3$.

i $V = h^3 - 4h + 9$ at $(2, 9)$

j $g(x) = x^4 - 2x^2 + 5x - 3$ at the point where $x = -1$.

3 Find the gradient of **i** the tangent and **ii** the normal to the curve:

a $y = x^2 + 1$ at $(3, 10)$

b $f(x) = 5 - x^2$ where $x = -4$

c $y = 2x^5 - 7x^2 + 4$ where $x = -1$

d $p(x) = x^6 - 3x^4 - 2x + 8$ where $x = 1$

e $f(x) = 4 - x - x^2$ at $(-6, 26)$

4 Find the equation of the tangent to the curve:

a $y = x^4 - 5x + 1$ at $(2, 7)$

b $f(x) = 5x^3 - 3x^2 - 2x + 6$ at $(1, 6)$

c $y = x^2 + 2x - 8$ at $(-3, -5)$

d $y = 3x^3 + 1$ where $x = 2$

e $v = 4t^4 - 7t^3 - 2$ where $t = 2$

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5 Find the equation of the normal to the curve:

a $f(x) = x^3 - 3x + 5$ at $(3, 23)$

b $y = x^2 - 4x - 5$ at $(-2, 7)$

c $f(x) = 7x - 2x^2$ where $x = 6$

d $y = 7x^2 - 3x - 3$ at $(-3, 69)$

e $y = x^4 - 2x^3 + 4x + 1$ where $x = 1$

6 Find the equation of **i** the tangent and **ii** the normal to the curve:

a $f(x) = 4x^2 - x + 8$ at $(1, 11)$

b $y = x^3 - 2x^2 - 5x$ at $(-3, -30)$

c $F(x) = x^5 - 5x^3$ where $x = 1$

d $y = x^2 - 8x + 7$ at $(3, -8)$.

7 For the curve $y = x^3 - 27x - 5$, find values of x for which $\frac{dy}{dx} = 0$.

8 Find the coordinates of the points at which the curve $y = x^3 + 1$ has a tangent with a gradient of 3.

9 A function $f(x) = x^2 + 4x - 12$ has a tangent with a gradient of -6 at point P on the curve. Find the coordinates of P.

10 The tangent at point P on the curve $y = 4x^2 + 1$ is parallel to the x-axis. Find the coordinates of P.

11 Find the coordinates of point Q where the tangent to the curve $y = 5x^2 - 3x$ is parallel to the line $7x - y + 3 = 0$.

12 Find the coordinates of point S where the tangent to the curve $y = x^2 + 4x - 1$ is perpendicular to the line $4x + 2y + 7 = 0$.

13 The curve $y = 3x^2 - 4$ has a gradient of 6 at point A.

a Find the coordinates of A.

b Find the equation of the tangent to the curve at A.

14 A function $h = 3t^2 - 2t + 5$ has a tangent at the point where $t = 2$. Find the equation of the tangent.

15 A function $f(x) = 2x^2 - 8x + 3$ has a tangent parallel to the line $4x - 2y + 1 = 0$ at point P. Find the equation of the tangent at P.

16 Find the equation of the tangent to the curve $y = \frac{1}{x^3}$ at $\left(2, \frac{1}{8}\right)$.

17 Find the equation of the tangent to $f(x) = 6\sqrt{x}$ at the point where $x = 9$.

18 Find the equation of the tangent to the curve $y = \frac{4}{x}$ at $\left(8, \frac{1}{2}\right)$.

19 If the gradient of the tangent to $y = \sqrt{x}$ is $\frac{1}{6}$ at point A, find the coordinates of A.

6.07 Chain rule

WS Chain rule

We looked at composite functions in Chapter 5, *Further functions*.

The **chain rule** is a method for differentiating composite functions. It is also called the **composite function** rule or the '**function of a function**' **rule**.

The chain rule

If a function y can be written as a composite function where $y = f(u(x))$, then:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

The chain rule

EXAMPLE 14

Differentiate:

a $y = (5x + 4)^7$ **b** $y = (3x^2 + 2x - 1)^9$ **c** $y = \sqrt{3 - x}$

Solution

a Let $u = 5x + 4$

Then $\dfrac{du}{dx} = 5$

$y = u^7$

$\therefore \dfrac{dy}{du} = 7u^6$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= 7u^6 \times 5 \\ &= 35u^6 \\ &= 35(5x + 4)^6\end{aligned}$$

b Let $u = 3x^2 + 2x - 1$

Then $\dfrac{du}{dx} = 6x + 2$

$y = u^9$

$\therefore \dfrac{dy}{du} = 9u^8$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= 9u^8 \times (6x + 2) \\ &= 9(3x^2 + 2x - 1)^8(6x + 2) \\ &= 9(6x + 2)(3x^2 + 2x - 1)^8\end{aligned}$$

c $y = \sqrt{3 - x} = (3 - x)^{\frac{1}{2}}$

Let $u = 3 - x$

Then $\dfrac{du}{dx} = -1$

$y = u^{\frac{1}{2}}$

$\dfrac{dy}{du} = \dfrac{1}{2}u^{-\frac{1}{2}}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= \frac{1}{2}u^{-\frac{1}{2}} \times (-1) \\ &= -\frac{1}{2}(3 - x)^{-\frac{1}{2}} \\ &= -\frac{1}{2\sqrt{3 - x}}\end{aligned}$$

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You might see a pattern when using the chain rule. The derivative of a composite function is the product of the derivatives of 2 functions.

The derivative of $[f(x)]^n$

$$\frac{d}{dx}[f(x)]^n = f'(x)n[f(x)]^{n-1}$$

EXAMPLE 15

Differentiate:

a $y = (8x^3 - 1)^5$ **b** $y = (3x + 8)^{11}$ **c** $y = \frac{1}{(6x+1)^2}$

Solution

a $\frac{dy}{dx} = f'(x) \times n[f(x)]^{n-1}$

$= 24x^2 \times 5(8x^3 - 1)^4$

$= 120x^2(8x^3 - 1)^4$

b $\frac{dy}{dx} = f'(x) \times n[f(x)]^{n-1}$

$= 3 \times 11(3x + 8)^{10}$

$= 33(3x + 8)^{10}$

c $y = \frac{1}{(6x+1)^2} = (6x+1)^{-2}$

$\frac{dy}{dx} = f'(x) \times n[f(x)]^{n-1}$

$= 6 \times (-2)(6x + 1)^{-3}$

$= -12(6x + 1)^{-3}$

$= -\frac{12}{(6x+1)^3}$

Exercise 6.07 Chain rule

1 Differentiate:

a $y = (x + 3)^4$ **b** $y = (2x - 1)^3$ **c** $y = (5x^2 - 4)^7$

d $y = (8x + 3)^6$ **e** $y = (1 - x)^5$ **f** $y = 3(5x + 9)^9$

g $y = 2(x - 4)^2$ **h** $y = (2x^3 + 3x)^4$ **i** $y = (x^2 + 5x - 1)^8$

j $y = (x^6 - 2x^2 + 3)^6$ **k** $y = (3x-1)^{\frac{1}{2}}$ **l** $y = (4 - x)^{-2}$

m $y = (x^2 - 9)^{-3}$ **n** $y = (5x+4)^{\frac{1}{3}}$ **o** $y = (x^3 - 7x^2 + x)^{\frac{3}{4}}$

p $y = \sqrt{3x+4}$

q $y = \dfrac{1}{5x-2}$

r $y = \dfrac{1}{(x^2+1)^4}$

s $y = \sqrt[3]{(7-3x)^2}$

t $y = \dfrac{5}{\sqrt{4+x}}$

u $y = \dfrac{1}{2\sqrt{3x-1}}$

v $y = \dfrac{3}{4(2x+7)^9}$

w $y = \dfrac{1}{x^4-3x^3+3x}$

x $y = \sqrt[3]{(4x+1)^4}$

y $y = \dfrac{1}{\sqrt[4]{(7-x)^5}}$

2 Find the gradient of the tangent to the curve $y = (3x-2)^3$ at the point (1, 1).

3 If $f(x) = 2(x^2-3)^5$, evaluate $f'(2)$.

4 The curve $y = \sqrt{x-3}$ has a tangent with gradient $\dfrac{1}{2}$ at point N. Find the coordinates of N.

5 For what values of x does the function $f(x) = \dfrac{1}{4x-1}$ have $f'(x) = -\dfrac{4}{49}$?

6 Find the equation of the tangent to $y = (2x+1)^4$ at the point where $x = -1$.

7 Find the equation of the tangent to the curve $y = (2x-1)^8$ at the point where $x = 1$.

8 Find the equation of the normal to the curve $y = (3x-4)^3$ at $(1, -1)$.

9 Find the equation of the normal to the curve $y = (x^2+1)^4$ at (1, 16).

10 Find the equation of **a** the tangent and **b** the normal to the curve $f(x) = \dfrac{1}{2x+3}$ at the point where $x = -1$.

WS

Product rule

6.08 Product rule

The **product rule** is a method for differentiating the product of 2 functions.

The product rule

If $y = uv$ where u and v are functions, then:

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$\text{or} \quad y' = u'v + v'u.$$

We can also write the product rule the other way round (differentiating v first), but the above formulas will also help us to remember the quotient rule in the next section.

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EXAMPLE 16

Differentiate:

a $y = (3x + 1)(x - 5)$ **b** $y = 9x^3(2x - 7)$

Solution

a You could expand the brackets and then differentiate:

$$\begin{aligned} y &= (3x + 1)(x - 5) \\ &= 3x^2 - 15x + x - 5 \\ &= 3x^2 - 14x - 5 \\ \frac{dy}{dx} &= 6x - 14 \end{aligned}$$

Using the product rule:

$y = uv$ where $u = 3x + 1$ and $v = x - 5$

$u' = 3$ $v' = 1$

$$\begin{aligned} y' &= u'v + v'u \\ &= 3(x - 5) + 1(3x + 1) \\ &= 3x - 15 + 3x + 1 \\ &= 6x - 14 \end{aligned}$$

b $y = uv$ where $u = 9x^3$ and $v = 2x - 7$

$u' = 27x^2$ $v' = 2$

$$\begin{aligned} y' &= u'v + v'u \\ &= 27x^2(2x - 7) + 2(9x^3) \\ &= 54x^3 - 189x^2 + 18x^3 \\ &= 72x^3 - 189x^2 \end{aligned}$$

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We can use the product rule together with the chain rule.

EXAMPLE 17

Differentiate:

a $y = 2x^5(5x+3)^3$ **b** $y = (3x-4)\sqrt{5-2x}$

Solution

a $y = uv$ where $u = 2x^5$ and $v = (5x+3)^3$

$u' = 10x^4$ $\quad v' = 5 \times 3(5x+3)^2$ using chain rule

$= 15(5x+3)^2$

$$\begin{aligned} y' &= u'v + v'u \\ &= 10x^4\,(5x+3)^3 + 15(5x+3)^2\,2x^5 \\ &= 10x^4(5x+3)^3 + 30x^5(5x+3)^2 \\ &= 10x^4(5x+3)^2[(5x+3)+3x] \\ &= 10x^4(5x+3)^2(8x+3) \end{aligned}$$

b $y = uv$ where $u = 3x-4$ and $v = \sqrt{5-2x} = (5-2x)^{\frac{1}{2}}$

$u' = 3$ $\quad v' = -2 \times \frac{1}{2}(5-2x)^{-\frac{1}{2}}$ using chain rule

$$\begin{aligned} &= -(5-2x)^{-\frac{1}{2}} \\ &= -\frac{1}{(5-2x)^{\frac{1}{2}}} \\ &= -\frac{1}{\sqrt{5-2x}} \end{aligned}$$

$$\begin{aligned} y' &= u'v + v'u \\ &= 3 \cdot \sqrt{5-2x} + -\frac{1}{\sqrt{5-2x}}(3x-4) \\ &= 3\sqrt{5-2x} - \frac{3x-4}{\sqrt{5-2x}} \\ &= \frac{3\sqrt{5-2x} \times \sqrt{5-2x}}{\sqrt{5-2x}} - \frac{3x-4}{\sqrt{5-2x}} \\ &= \frac{3(5-2x)}{\sqrt{5-2x}} - \frac{3x-4}{\sqrt{5-2x}} \end{aligned}$$

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$$= \frac{3(5-2x)-(3x-4)}{\sqrt{5-2x}}$$

$$= \frac{15-6x-3x+4}{\sqrt{5-2x}}$$

$$= \frac{19-9x}{\sqrt{5-2x}}$$

Exercise 6.08 Product rule

1 Differentiate:

a $y = x^3(2x+3)$ **b** $y = (3x-2)(2x+1)$ **c** $y = 3x(5x+7)$

d $y = 4x^4(3x^2-1)$ **e** $y = 2x(3x^4-x)$ **f** $y = x^2(x+1)^3$

g $y = 4x(3x-2)^5$ **h** $y = 3x^4(4-x)^3$ **i** $y = (x+1)(2x+5)^4$

2 Find the gradient of the tangent to the curve $y = 2x(3x-2)^4$ at (1, 2).

3 If $f(x) = (2x+3)(3x-1)^5$, evaluate $f'(1)$.

4 Find the exact gradient of the tangent to the curve $y = x\sqrt{2x+5}$ at the point where $x = 1$.

5 Find the gradient of the tangent where $t = 3$ given $x = (2t-5)(t+1)^3$.

6 Find the equation of the tangent to the curve $y = x^2(2x-1)^4$ at (1, 1).

7 Find the equation of the tangent to $h = (t+1)^2(t-1)^7$ at (2, 9).

8 Find exact values of x for which the gradient of the tangent to the curve $y = 2x(x+3)^2$ is 14.

9 Given $f(x) = (4x-1)(3x+2)^2$, find the equation of the tangent at the point where $x = -1$.

WS Quotient rule

6.09 Quotient rule

The **quotient rule** is a method for differentiating the ratio of 2 functions.

WS Rules of differentiation

WS Mixed differentiation problems

The quotient rule

If $y = \frac{u}{v}$ where u and v are functions, then:

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\text{or } y' = \frac{u'v - v'u}{v^2}.$$

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EXAMPLE 18

Differentiate:

a $y = \dfrac{3x-5}{5x+2}$ **b** $y = \dfrac{4x^3-5x+2}{x^3-1}$

Solution

a $y = \dfrac{u}{v}$ where $u = 3x - 5$ and $v = 5x + 2$

$u' = 3$ $v' = 5$

$$\begin{aligned} y' &= \frac{u'v - v'u}{v^2} \\ &= \frac{3(5x+2)-5(3x-5)}{(5x+2)^2} \\ &= \frac{15x+6-15x+25}{(5x+2)^2} \\ &= \frac{31}{(5x+2)^2} \end{aligned}$$

b $y = \dfrac{u}{v}$ where $u = 4x^3 - 5x + 2$ and $v = x^3 - 1$

$u' = 12x^2 - 5$ $v' = 3x^2$

$$\begin{aligned} y' &= \frac{u'v - v'u}{v^2} \\ &= \frac{(12x^2-5)(x^3-1)-3x^2(4x^3-5x+2)}{(x^3-1)^2} \\ &= \frac{12x^5-12x^2-5x^3+5-12x^5+15x^3-6x^2}{(x^3-1)^2} \\ &= \frac{10x^3-18x^2+5}{(x^3-1)^2} \end{aligned}$$

Exercise 6.09 Quotient rule

1 Differentiate:

a $y = \dfrac{1}{2x-1}$ **b** $y = \dfrac{3x}{x+5}$ **c** $y = \dfrac{x^3}{x^2-4}$ **d** $y = \dfrac{x-3}{5x+1}$

e $y = \dfrac{x-7}{x^2}$ **f** $y = \dfrac{5x+4}{x+3}$ **g** $y = \dfrac{x}{2x^2-1}$ **h** $y = \dfrac{x+4}{x-2}$

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i $y = \dfrac{2x+7}{4x-3}$ **j** $y = \dfrac{x+5}{3x+1}$ **k** $y = \dfrac{x+1}{3x^2-7}$ **l** $y = \dfrac{2x^2}{2x-3}$

m $y = \dfrac{x^2+4}{x^2-5}$ **n** $y = \dfrac{x^3}{x+4}$ **o** $y = \dfrac{x^3+2x-1}{x+3}$ **p** $y = \dfrac{x^2-2x-1}{3x+4}$

q $y = \dfrac{2x}{(x+5)^{\frac{1}{2}}}$ **r** $y = \dfrac{x-1}{(7x+2)^4}$ **s** $y = \dfrac{3x+1}{\sqrt{x+1}}$ **t** $y = \dfrac{\sqrt{x-1}}{2x-3}$

2 Find the gradient of the tangent to the curve $y = \dfrac{2x}{3x+1}$ at $\left(1, \dfrac{1}{2}\right)$.

3 If $f(x) = \dfrac{4x+5}{2x-1}$, evaluate $f'(2)$.

4 Find values of x for which the gradient of the tangent to $y = \dfrac{4x-1}{2x-1}$ is -2.

5 Given $f(x) = \dfrac{2x}{x+3}$, find x if $f'(x) = \dfrac{1}{6}$.

6 Find the equation of the tangent to the curve $y = \dfrac{x}{x+2}$ at $\left(4, \dfrac{2}{3}\right)$.

7 Find the equation of the tangent to the curve $y = \dfrac{x^2-1}{x+3}$ at $x = 2$.

6.10 Rates of change

WS Rates of change

WS Instantaneous rates of change

WS Graphs of rates of change

We know that the gradient $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ of the secant passing through 2 points on the graph of a function gives the **average rate of change** between those 2 points.

Now consider a quantity Q that changes with time, giving the function $Q(t)$.

Average rate of change

The average rate of change of a quantity Q with respect to time t is $\dfrac{Q_2 - Q_1}{t_2 - t_1}$.

We know that the gradient $\dfrac{dy}{dx}$ of the tangent at a point on the graph of a function gives the **instantaneous rate of change** at that point.

Instantaneous rate of change

The instantaneous rate of change of a quantity Q with respect to time t is $\dfrac{dQ}{dt}$.

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EXAMPLE 19

a The number of bacteria in a culture increases according to the function $B = 2t^4 - t^2 + 2000$, where t is time in hours. Find:

i the number of bacteria initially

ii the average rate of change in number of bacteria between 2 and 3 hours

iii the number of bacteria after 5 hours

iv the rate at which the number of bacteria is increasing after 5 hours.

b An object travels a distance according to the function $D = t^2 + t + 5$, where D is in metres and t is in seconds. Find the speed at which it is travelling at:

i 4 s **ii** 10 s

Solution

a **i** $B = 2t^4 - t^2 + 2000$

Initially, $t = 0$:

$$B = 2(0)^4 - (0)^2 + 2000$$
$$= 2000$$

So there are 2000 bacteria initially.

ii When $t = 2$, $B = 2(2)^4 - (2)^2 + 2000$

$$= 2028$$

When $t = 3$, $B = 2(3)^4 - (3)^2 + 2000$

$$= 2153$$

$$\text{Average rate of change} = \frac{B_2 - B_1}{t_2 - t_1}$$
$$= \frac{2153 - 2028}{3 - 2}$$
$$= 125 \text{ bacteria/hour}$$

So the average rate of change is 125 bacteria per hour.

iii When $t = 5$, $B = 2(5)^4 - (5)^2 + 2000$

$$= 3225$$

So there will be 3225 bacteria after 5 hours.

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iv The instantaneous rate of change is given by the derivative $\frac{dB}{dt} = 8t^3 - 2t$.

When $t = 5$, $\frac{dB}{dt} = 8(5)^3 - 2(5)$

$= 990$

So the rate of increase after 5 hours will be 990 bacteria per hour.

b Speed is the rate of change of distance over time: $\frac{dD}{dt} = 2t + 1$.

i When $t = 4$, $\frac{dD}{dt} = 2(4) + 1$

$= 9$

So speed after 4 s is 9 m/s.

ii When $t = 10$, $\frac{dD}{dt} = 2(10) + 1$

$= 21$

So speed after 10 s is 21 m/s.

Displacement, velocity and acceleration

Displacement (x) measures the distance of an object from a fixed point (origin). It can be positive or negative or 0, according to where the object is.

Velocity (v) is the rate of change of displacement with respect to time, and involves speed and direction.

Velocity

Velocity $v = \frac{dx}{dt}$ is the instantaneous rate of change of displacement x over time t.

Acceleration (a) is the rate of change of velocity with respect to time.

Acceleration

Acceleration $a = \frac{dv}{dt}$ is the instantaneous rate of change of velocity v over time t.

We usually write velocity units as km/h or m/s, but we can also use index notation and write km h^{-1} or m s^{-1}.

With acceleration units, we write km/h/h as km/h^2, or in index notation we write km h^{-2}.

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EXAMPLE 20

A ball rolls down a ramp so that its displacement x cm in t seconds is $x = 16 - t^2$.

a Find its initial displacement.

b Find its displacement at 3 s.

c Find its velocity at 2 s.

d Show that the ball has a constant acceleration of -2 cm s^{-2}.

Solution

a $x = 16 - t^2$

Initially, $t = 0$:

$x = 16 - 0^2$

$\quad = 16$

So the ball's initial displacement is 16 cm.

b When $t = 3$:

$x = 16 - 3^2$

$\quad = 7$

So the ball's displacement at 3 s is 7 cm.

c $v = \dfrac{dx}{dt}$

$\quad = -2t$

When $t = 2$:

$v = -2(2)$

$\quad = -4$

So the ball's velocity at 2 s is -4 cm s^{-1}.

> x is measured in cm, t is measured in s, so v is measured in cm/s or cm s^{-1}.

d $a = \dfrac{dv}{dt}$

$\quad = -2$

So acceleration is constant at -2 cm s^{-2}.

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Exercise 6.10 Rates of change

1 Find the formula for the rate of change for each function.

a $h = 20t - 4t^2$ **b** $D = 5t^3 + 2t^2 + 1$ **c** $A = 16x - 2x^2$

d $x = 3t^5 - t^4 + 2t - 3$ **e** $V = \frac{4}{3}\circ r^3$ **f** $S = 2\pi r + \frac{50}{r^2}$

g $D = \sqrt{x^2 - 4}$ **h** $S = 800r + \frac{400}{r}$

2 If $h = t^3 - 7t + 5$, find:

a the average rate of change of h between $t = 3$ and $t = 4$

b the instantaneous rate of change of h when $t = 3$.

3 The volume of water V in litres flowing through a pipe after t seconds is given by $V = t^2 + 3t$. Find the rate at which the water is flowing when $t = 5$.

4 The mass in grams of a melting ice block is given by the formula $M = t - 2t^2 + 100$, where t is time in minutes.

a Find the average rate of change at which the ice block is melting between:

i 1 and 3 minutes **ii** 2 and 5 minutes.

b Find the rate at which it will be melting at 5 minutes.

5 The surface area in cm^2 of a balloon being inflated is given by $S = t^3 - 2t^2 + 5t + 2$, where t is time in seconds. Find the rate of increase in the balloon's surface area at 8 s.

6 A circular disc expands as it is heated. The area, in cm^2, of the disc increases according to the formula $A = 4t^2 + t$, where t is time in minutes. Find the rate of increase in the area after 5 minutes.

7 A car is d km from home after t hours according to the formula $d = 10t^2 + 5t + 11$.

a How far is the car from home:

i initially? **ii** after 3 hours? **iii** after 5 hours?

b At what speed is the car travelling after:

i 3 hours? **ii** 5 hours?

8 According to Boyle's Law, the pressure of a gas is given by the formula $P = \frac{k}{V}$ where k is a constant and V is the volume of the gas. If $k = 100$ for a certain gas, find the rate of change in the pressure when $V = 20$.

9 The displacement of a particle is $x = t^3 - 9t$ cm, where t is time in seconds.

a Find the velocity of the particle at 3 s.

b Find the acceleration at 2 s.

c Show that the particle is initially at the origin, and find any other times that the particle will be at the origin.

d At what time will the acceleration be 30 cm s^{-2}?

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10 A particle is moving with displacement $s = 2t^2 - 8t + 3$, where s is in metres and t is in seconds.

a Find its initial velocity.

b Show that its acceleration is constant and find its value.

c Find its displacement at 5 s.

d Find when the particle's velocity is zero.

e What will the particle's displacement be at that time?

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6. TEST YOURSELF

Practice quiz

Derivatives find-a-word

For Questions **1** to **4**, select the correct answer **A**, **B**, **C** or **D**.

1 Find the derivative of $\dfrac{2}{3x^4}$.

A $\dfrac{8}{3x^5}$ **B** $-\dfrac{8}{3x^3}$ **C** $-\dfrac{8}{3x^5}$ **D** $\dfrac{8}{3x^3}$

2 Differentiate $3x(x^3 - 5)$.

A $4x^3$ **B** $12x^3 - 15$ **C** $9x^2$ **D** $3x^4 - 15x$

3 The derivative of $y = f(x)$ is given by:

A $\lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{x-h}$ **B** $\lim\limits_{x\to 0} \dfrac{f(x+h)-f(x)}{x}$

C $\lim\limits_{h\to 0} \dfrac{f(x)-f(x+h)}{h}$ **D** $\lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

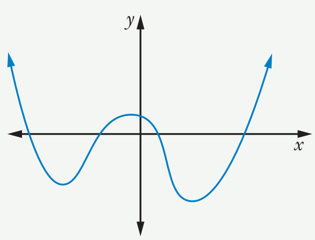
4 Which of the following is the chain rule (there is more than one answer)?

A $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{dx}{du}$ **B** $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

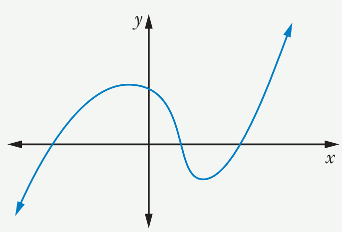
C $\dfrac{dy}{dx} = nf'(x)f(x)^{n-1}$ **D** $\dfrac{dy}{dx} = nf(x)^{n-1}$

5 Sketch the derivative function of each graph.

a



b



6 Differentiate $y = 5x^2 - 3x + 2$ from first principles.

7 Differentiate:

a $y = 7x^6 - 3x^3 + x^2 - 8x - 4$ **b** $y = 3x^{-4}$ **c** $y = \dfrac{2}{(x+1)^4}$

d $y = x^2\sqrt{x}$ **e** $y = (x^2 + 4x - 2)^9$ **f** $y = \dfrac{3x-2}{2x+1}$

g $y = x^3(3x+1)^6$

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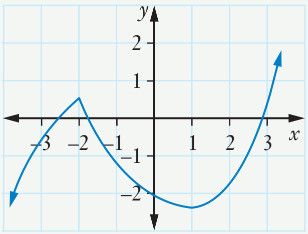
8 Find $\dfrac{dv}{dt}$ if $v = 2t^2 - 3t - 4$.

9 Find the gradient of the tangent to the curve $y = x^3 + 3x^2 + x - 5$ at $(1, 0)$.

10 If $h = 60t - 3t^2$, find $\dfrac{dh}{dt}$ when $t = 3$.

11 For each graph of a function, find all values of x where it is not differentiable.

a



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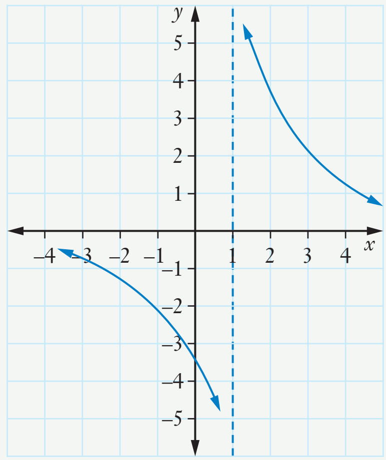
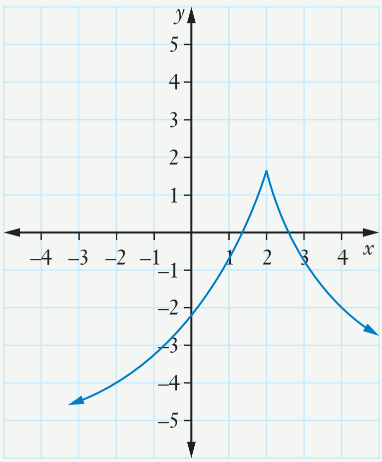


c



12 Differentiate:

a $y = \dfrac{4}{x}$

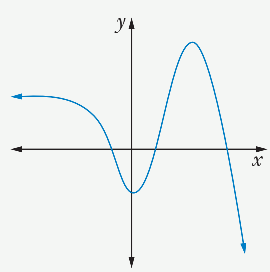
b $f(x) = \sqrt[5]{x}$

c $f(x) = 2(4x + 9)^4$

d $y = (3x + 2)(x - 1)^3$

e $f(x) = \dfrac{x^3 - 3}{2x + 5}$

13 Sketch the derivative function of this curve.



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14 Find the equation of the tangent to the curve $y = x^2 + 5x - 3$ at $(2, 11)$.

15 Find the point on the curve $y = x^2 - x + 1$ at which the tangent has a gradient of 3.

16 Find $\frac{dS}{dr}$ if $S = 4\pi r^2$.

17 Find the gradient of the secant on the curve $f(x) = x^2 - 3x + 1$ between the points where $x = 1$ and $x = 1.1$.

18 At which points on the curve $y = 2x^3 - 9x^2 - 60x + 3$ are the tangents horizontal?

19 Find the equation of the tangent to the curve $y = x^2 + 2x - 5$ that is parallel to the line $y = 4x - 1$.

20 **a** Differentiate $s = ut + \frac{1}{2}at^2$ with respect to t.

b Find the value of t for which $\frac{ds}{dt} = 5$, $u = 7$ and $a = -10$.

21 Find the equation of the tangent to the curve $y = \frac{1}{3x}$ at the point where $x = \frac{1}{6}$.

22 A ball is thrown into the air and its height h metres over t seconds is given by $h = 4t - t^2$.

a Find the height of the ball:

i initially **ii** at 2 s **iii** at 3 s **iv** at 3.5 s

b Find the average rate of change of the height between:

i 1 and 2 seconds **ii** 2 and 3 seconds

c Find the rate at which the ball is moving:

i initially **ii** at 2 s **iii** at 3 s

23 If $f(x) = x^2 - 3x + 5$, find:

a $f(x + h)$ **b** $f(x + h) - f(x)$ **c** $f'(x)$

24 Given $f(x) = (4x - 3)^5$, find the value of:

a $f(1)$ **b** $f'(1)$

25 Find $f'(4)$ when $f(x) = (x - 3)^9$.

26 Differentiate:

a $y = 3(x^2 - 6x + 1)^4$ **b** $y = \frac{2}{\sqrt{3x - 1}}$

27 A particle moves so that its displacement after t seconds is $x = 4t^2 - 5t^3$ metres. Find:

a its initial displacement, velocity and acceleration

b when $x = 0$

c its velocity and acceleration at 2 s.

6. CHALLENGE EXERCISE

1 Find the equations of the tangents to the curve $y = x(x - 1)(x + 2)$ at the points where the curve cuts the x-axis.

2 **a** Find the points on the curve $y = x^3 - 6$ where the tangents are parallel to the line $y = 12x - 1$.

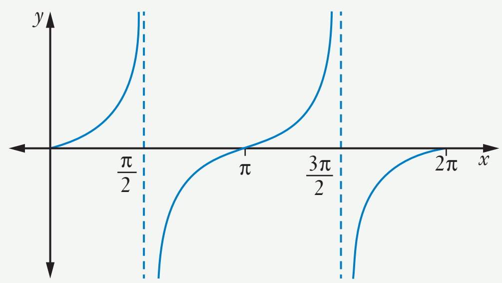
b Hence find the equations of the normals to the curve at those points.

3 The normal to the curve $y = x^2 + 1$ at the point where $x = 2$ cuts the curve again at point P. Find the coordinates of P.

4 The equation of the tangent to the curve $y = x^4 - nx^2 + 3x - 2$ at the point where $x = -2$ is given by $3x - y - 2 = 0$. Evaluate n.

5 **a** Find any points at which the graphed function is not differentiable.

b Sketch the derivative function for the graph.



6 Find the exact gradient of the tangent to the curve $y = \sqrt{x^2 - 3}$ at the point where $x = 5$.

7 Find the equation of the normal to the curve $y = 3\sqrt{x + 1}$ at the point where $x = 8$.

8 **a** Find the equations of the tangents to the parabola $y = 2x^2$ at the points where the line $6x - 8y + 1 = 0$ intersects with the parabola.

b Show that the tangents are perpendicular.

9 Find any x values of the function $f(x) = \dfrac{2}{x^3 - 8x^2 + 12x}$ where it is not differentiable.

10 Find the equation of the chord joining the points of contact of the tangents to the curve $y = x^2 - x - 4$ with gradients 3 and -1.

11 For the function $f(x) = ax^2 + bx + c$, $f(2) = 4$, $f'(1) = 0$ and $f'(-3) = 8$. Evaluate a, b and c.

12 For the function $f(x) = x^3$:

a Show that $f(x + h) = x^3 + 3x^2h + 3xh^2 + h^3$.

b Show that $f'(x) = 3x^2$ by differentiating from first principles.

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13 Consider the function $f(x) = \frac{1}{x}$.

a Find the gradient of the secant between:

i $f(1)$ and $f(1.1)$ **ii** $f(1)$ and $f(1.01)$ **iii** $f(1)$ and $f(0.99)$

b Estimate the gradient of the tangent to the curve at the point where $x = 1$.

c Show that $\frac{1}{x+h} - \frac{1}{x} = \frac{-h}{x(x+h)}$.

d Hence show that $f'(x) = -\frac{1}{x^2}$ by differentiating from first principles.

14 The displacement of a particle is given by $x = (t^3 + 1)^6$, where x is in metres and t is in seconds.

a Find the initial displacement and velocity of the particle.

b Find its acceleration after 2 s in scientific notation, correct to 3 significant figures.

c Show that the particle is never at the origin.

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