8.

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

In this chapter you will study the definition and laws of logarithms and their relationship with the exponential and logarithmic functions. You will meet a new irrational number, *e*, that has special properties, solve exponential and logarithmic equations, and examine applications of exponential and logarithmic functions.

CHAPTER OUTLINE

IN THIS CHAPTER YOU WILL:

- graph exponential and logarithmic functions
- understand and use Euler's number, *e*
- differentiate exponential functions
- convert between exponential and logarithmic forms using the definition of a logarithm
- identify and apply logarithm laws
- solve exponential equations using logarithms
- solve practical formulas involving exponents and logarithms

TERMINOLOGY

Euler's number: This number, e, approximately 2.718 28, is an important constant that is the base of natural logarithms

exponential function: A function in the form $y = a^x$

logarithm: The logarithm of a positive number y is the power to which a given number a, called the base, must be raised in order to produce the number y, so $\log_a y = x$ means $y = a^x$

logarithmic function: A function in the form $y = \log_a x$

WS Graphing exponentials

WS Exponential functions

WS Translating exponential graphs

8.01 Exponential functions

An **exponential function** is in the form $y = a^x$, where $a > 0$.

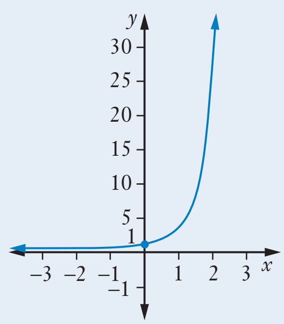
EXAMPLE 1

Sketch the graph of the function $y = 5^x$ and state its domain and range.

Solution

Complete a table of values for $y = 5^x$.

x	-3	-2	-1	0	1	2	3
y	$\frac{1}{125}$	$\frac{1}{25}$	$\frac{1}{5}$	1	5	25	125



Notice that a^x is always positive. So there is no x-intercept and $y > 0$.

For the y-intercept, when $x = 0$, $y = 5^0 = 1$.

The y-intercept is 1.

From the graph, the domain is $(-\infty, \infty)$ and the range is $(0, \infty)$.

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INVESTIGATION

THE VALUE OF a IN $y = a^x$

Notice that the exponential function $y = a^x$ is only defined for $a > 0$.

1 Suppose $a = 0$. What would the function $y = 0^x$ look like? Try completing a table of values or use technology to sketch the graph. Is the function defined for positive values of x, negative values of x or when $x = 0$? What if x is a fraction?

2 Suppose $a < 0$. What would the function $y = (-2)^x$ look like?

3 For $y = 0^x$ and $y = (-2)^x$:

a is it possible to graph these functions at all?

b are there any discontinuities on the graphs?

c do they have a domain and range?

The exponential function $y = a^x$

- Domain $(-\infty, \infty)$, range $(0, \infty)$.
- The y-intercept ($x = 0$) is always 1 because $a^0 = 1$.
- The graph is always above the x-axis and there is no x-intercept ($y = 0$) because $a^x > 0$ for all values of x.
- The x-axis is an **asymptote**.

EXAMPLE 2

Sketch the graph of:

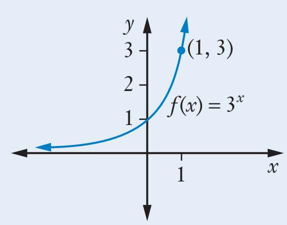
a $f(x) = 3^x$

b $y = 2^x + 1$

Solution

a The curve is above the x-axis with y-intercept 1. We must show another point on the curve.

$f(1) = 3^1 = 3$.



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b For y-intercept, $x = 0$:

$y = 2^0 + 1$

$= 1 + 1$

$= 2$

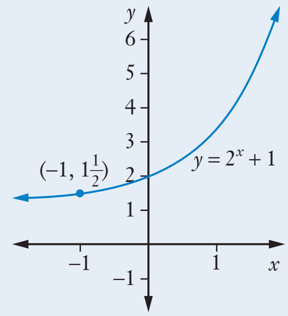
Find another point, when $x = -1$:

$y = 2^{-1} + 1$

$= \frac{1}{2} + 1$

$= 1\frac{1}{2}$

Notice that this is the graph of $y = 2^x$ moved up 1 unit.



EXAMPLE 3

Sketch the graph of:

a $f(x) = 3(4^x)$ **b** $y = 2^{x+1}$

Solution

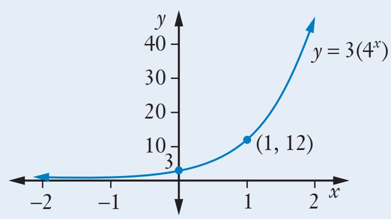
a The values of $f(x) = 3(4^x)$ will be 3 times greater than 4^x so its curve will be steeper.

$f(-1) = 3(4^{-1}) = 0.75$

$f(0) = 3(4^0) = 3$

$f(1) = 3(4^1) = 12$

$f(2) = 3(4^2) = 48$

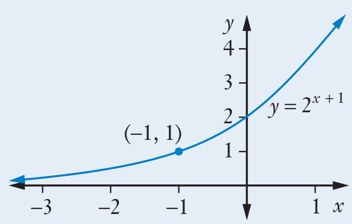


b $f(-1) = 2^{-1+1} = 1$

$f(0) = 2^{0+1} = 2$

$f(1) = 2^{1+1} = 4$

$f(2) = 2^{2+1} = 8$



This is the graph of $y = 2^x$ shifted 1 unit left.

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Reflections of exponential functions

We can reflect the graph of $y = a^x$ using what we learned in Chapter 5, *Further functions*.

EXAMPLE 4

Given $f(x) = 3^x$, sketch the graph of:

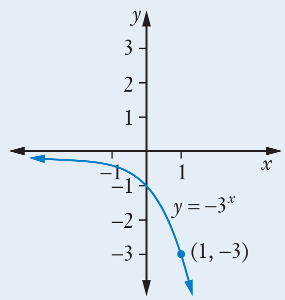
a $y = -3^x$ **b** $y = 3^{-x}$ **c** $y = -3^{-x}$

Solution

a Given $f(x) = 3^x$, then $y = -f(x) = -3^x$.

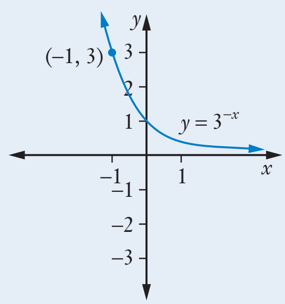
This is a reflection of $f(x)$ in the x-axis.

Note: -3^x means $-(3^x)$, *not* $(-3)^x$.



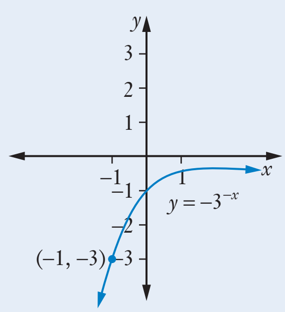
b Given $f(x) = 3^x$, then $y = f(-x) = 3^{-x}$.

This is a reflection of $f(x)$ in the y-axis.



c Given $f(x) = 3^x$, then $y = -f(-x) = -3^{-x}$.

This is a reflection of $f(x)$ in both the x- and y-axes.



INVESTIGATION

GRAPHS OF EXPONENTIAL FUNCTIONS

Use a graphics calculator or graphing software to sketch the graphs of the exponential functions below. Look for similarities and differences within each set.

a $y = 2^x, y = 2^x + 1, y = 2^x + 3, y = 2^x - 5$

b $y = 3(2^x), y = 4(2^x), y = -2^x, y = -3(2^x)$

c $y = 3(2^x) + 1, y = 4(2^x) + 3, y = -2^x + 1, y = -3(2^x) - 3$

d $y = 2^{x+1}, y = 2^{x+2}, y = 2^{x-1}, y = 2^{x-3}, y = 2^{-x}$

e $y = 2^{-x}, y = 2\left(2^{-x}\right), y = -2^{-x}, y = -3\left(2^{-x}\right), y = 2^{-x-1}$

Exercise 8.01 Exponential functions

1 Sketch each exponential function.

a $y = 2^x$ **b** $y = 4^x$ **c** $f(x) = 3^x + 2$ **d** $y = 2^x - 1$

e $f(x) = 3(2^x)$ **f** $y = 4^{x+1}$ **g** $y = 3(4^{2x}) - 1$ **h** $f(x) = -2^x$

i $y = 2(4^{-x})$ **j** $f(x) = -3(5^{-x}) + 4$

2 State the domain and range of each function.

a $f(x) = 2^x$ **b** $y = 3^x + 5$ **c** $f(x) = 10^{-x}$ **d** $f(x) = -5^x + 1$

3 Given $f(x) = 2^x$ and $g(x) = 3x - 4$, find:

a $f(g(x))$ **b** $g(f(x))$

4 **a** Sketch the graph of $f(x) = 4(3^x) + 1$.

b Sketch the graph of:

i $y = f(-x)$ **ii** $y = -f(x)$ **iii** $y = -f(-x)$

5 Sales numbers, N, of a new solar battery are growing over t years according to the formula $N = 450(3^{0.9t})$.

a Draw a graph of this function.

b Find the initial number of sales when $t = 0$.

c Find the number of sales after:

i 3 years

ii 5 years

iii 10 years

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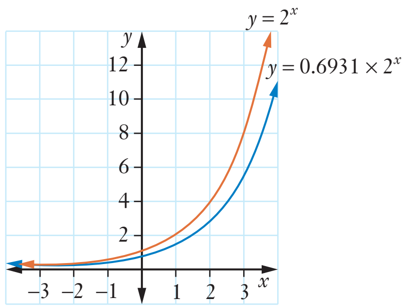
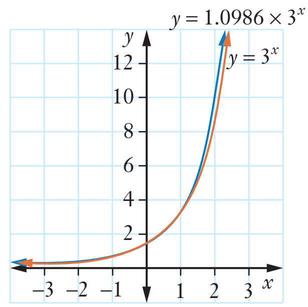
8.02 Euler's number, e

The gradient function of exponential functions is interesting. Notice that the gradient of an exponential function is always increasing, and increases at an increasing rate.

If you sketch the derivative function of an exponential function, then it too is an exponential function! Here are the graphs of the derivative functions (in blue) of $y = 2^x$ and $y = 3^x$ (in red) together with their equations.





Notice that the graph of the derivative function of $y = 3^x$ is very close to the graph of the original function.

We can find a number close to 3 that gives exactly the same derivative function as the original graph. This number is approximately 2.718 28, and is called **Euler's number**, e. Like π, the number e is irrational.

Euler's number

$$e \approx 2.718\ 28$$

DID YOU KNOW?

Leonhard Euler

Like π, Euler's number, e, is a **transcendental** number, which is an irrational number that is not a surd. This was proven by a French mathematician, **Charles Hermite**, in 1873. The Swiss mathematician **Leonhard Euler** (1707–83) gave e its symbol, and he gave an approximation of e to 23 decimal places. Now e has been calculated to over a trillion decimal places.

Euler gave mathematics much of its important notation. He caused π to become standard notation for pi and used i for the square root of -1. He also introduced the symbol Σ for sums and $f(x)$ notation for functions.

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EXAMPLE 5

Sketch the graph of the exponential function $y = e^x$.

Solution

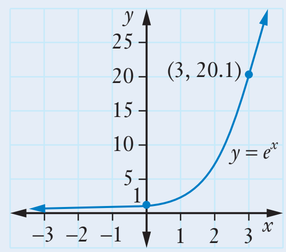
Use e^x on your calculator to draw up a table of values. For example, to calculate e^{-3}:

Casio scientific	Sharp scientific
SHIFT e^x (−) 3 =	2ndF e^x +/− 3 =

$e^{-3} = 0.049\,78\ldots$

x	−3	−2	−1	0	1	2	3
y	0.05	0.1	0.4	1	2.7	7.4	20.1

(rounded figures)



EXAMPLE 6

The salmon population in a river over time can be described by the exponential function $P = 200e^{0.3t}$ where t is time in years.

a Find the population after 3 years.

b Draw the graph of the population.

Solution

a $P = 200e^{0.3t}$

When $t = 3$:

$P = 200e^{0.3 \times 3}$

$= 491.9206\ldots$

≈ 492

So after 3 years there are 492 salmon.

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b The graph is an exponential curve. Finding some points will help us graph it accurately.

When $t = 0$: $P = 200e^{0.3 \times 0}$

$= 200$ This is also the P-intercept.

When $t = 1$: $P = 200e^{0.3 \times 1}$

$= 269.9717\ldots$

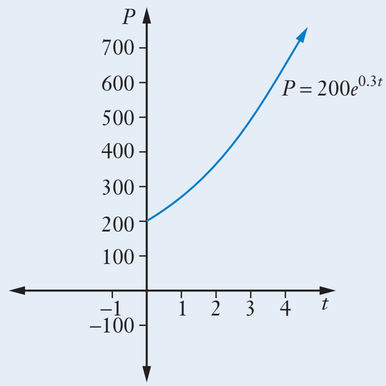
≈ 270

When $t = 2$: $P = 200e^{0.3 \times 2}$

$= 364.4237\ldots$

≈ 364

We already know $P \approx 492$ when $t = 3$.



Time, $t \geq 0$, so don't sketch the curve for negative values of t.

Exercise 8.02 Euler's number, *e*

1 Sketch the curve $f(x) = 2e^{x-2}$.

2 Evaluate, correct to 2 decimal places:

a $e^{1.5}$ **b** e^{-2} **c** $2e^{0.3}$ **d** $\dfrac{1}{e^3}$ **e** $-3e^{-3.1}$

3 Sketch each exponential function.

a $y = 2e^x$ **b** $f(x) = e^x + 1$ **c** $y = -e^x$ **d** $y = e^{-x}$ **e** $y = -e^{-x}$

4 State the domain and range of $f(x) = e^x - 2$.

5 If $f(x) = e^x$ and $g(x) = x^3 + 3$, find:

a $f(g(x))$ **b** $g(f(x))$

6 The volume V of a metal in mm^3 expands as it is heated over time according to the formula $V = 25e^{0.7t}$, where t is in minutes.

a Sketch the graph of $V = 25e^{0.7t}$.

b Find the volume of the metal at:

i 3 minutes **ii** 8 minutes

c Is this formula a good model for the rise in volume? Why?

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7 The mass of a radioactive substance in g is given by $M = 150e^{-0.014t}$ where t is in years. Find the mass after:

a 10 years **b** 50 years **c** 250 years

8 The number of koalas in a forest is declining according to the formula $N = 873e^{-0.078t}$ where t is the time in years.

a Sketch a graph showing this decline in numbers of koalas for the first 6 years.

b Find the number of koalas:

i initially **ii** after 5 years **iii** after 10 years

Photo courtesy Margaret Grove

9 An object is cooling down according to the exponential function $T = 23 + 125e^{-0.06t}$ where T is the temperature in $°C$ and t is time in minutes.

a Find the initial temperature.

b Find the temperature at:

i 2 minutes **ii** 5 minutes **iii** 10 minutes **iv** 2 hours

c What temperature is the object tending towards? Can you explain why?

10 A population is growing exponentially. If the initial population is 20 000 and after 5 years the population is 80 000, draw a graph showing this information.

11 The temperature of a piece of iron in a smelter is 1000°C and it is cooling down exponentially. After 10 minutes the temperature is 650°C. Draw a graph showing this information.

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8.03 Differentiation of exponential functions

WS
Differentiating exponential functions

Euler's number, e, is the special number such that the derivative function of $y = e^x$ is itself. The derivative of e^x is e^x.

Derivative of e^x

$$\frac{d}{dx}(e^x) = e^x$$

EXAMPLE 7

a Differentiate $y = e^x - 5x^2$.

b Find the equation of the tangent to the curve $y = e^x$ at the point $(1, e)$.

Solution

a $\frac{dy}{dx} = e^x - 10x$

b Gradient of the tangent:

$$\frac{dy}{dx} = e^x$$

At $(1, e)$:

$$\frac{dy}{dx} = e^1$$
$$= e$$

So $m = e$

Equation:

$$y - y_1 = m(x - x_1)$$
$$y - e = e(x - 1)$$
$$= ex - e$$
$$y = ex$$

The rule for differentiating $kf(x)$ works with the rule for e^x as well.

Derivative of ke^x

$$\frac{d}{dx}(ke^x) = ke^x$$

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EXAMPLE 8

a Differentiate $y = 5e^x$.

b Find the gradient of the normal to the curve $y = 3e^x$ at the point (0, 3).

Solution

a $\dfrac{dy}{dx} = 5e^x$

b Gradient of tangent:

$$\frac{dy}{dx} = 3e^x$$

At (0, 3):

$$\begin{aligned}\frac{dy}{dx} &= 3e^0 \\ &= 3 \text{ since } e^0 = 1\end{aligned}$$

So $m_1 = 3$

For normal:

$$\begin{aligned}m_1 m_2 &= -1 \\ 3m_2 &= -1 \\ m_2 &= -\frac{1}{3}\end{aligned}$$

So the gradient of the normal at (0, 3) is $-\dfrac{1}{3}$.

We can also use other differentiation rules such as the chain rule, product rule and quotient rule with the exponential function.

EXAMPLE 9

Differentiate:

a $y = e^{9x}$

b $y = e^{-5x}$

Solution

a Let $u = 9x$

Then $\dfrac{du}{dx} = 9$

$$\begin{aligned}y &= e^u \\ \frac{dy}{du} &= e^u \\ \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= e^u \times 9 \\ &= 9e^u \\ &= 9e^{9x}\end{aligned}$$

b Let $u = -5x$

Then $\dfrac{du}{dx} = -5$

$$\begin{aligned}y &= e^u \\ \frac{dy}{du} &= e^u \\ \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= e^u \times (-5) \\ &= -5e^u \\ &= -5e^{-5x}\end{aligned}$$

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The derivative of e^{ax}

$$\frac{d}{dx}(e^{ax}) = ae^{ax}$$

Proof

Let $u = ax$

Then $\frac{du}{dx} = a$

$$y = e^u$$

$$\frac{dy}{du} = e^u$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= e^u \times a \\ &= ae^u \\ &= ae^{ax}\end{aligned}$$

EXAMPLE 10

Differentiate:

a $y = (1 + e^x)^3$

b $y = \frac{2x+3}{e^x}$

Solution

a $$\begin{aligned}\frac{dy}{dx} &= 3(1 + e^x)^2 \times e^x \\ &= 3e^x(1 + e^x)^2\end{aligned}$$

b $$\begin{aligned}\frac{dy}{dx} &= \frac{u'v - v'u}{v^2} \\ &= \frac{2e^x - e^x(2x+3)}{(e^x)^2} \\ &= \frac{2e^x - 2xe^x - 3e^x}{e^{2x}} \\ &= \frac{-e^x - 2xe^x}{e^{2x}} \\ &= \frac{-e^x(1+2x)}{e^{2x}} \\ &= \frac{-(1+2x)}{e^x}\end{aligned}$$

Exercise 8.03 Differentiation of exponential functions

1 Differentiate:

a $y = 9e^x$ **b** $y = -e^x$ **c** $y = e^x + x^2$

d $y = 2x^3 - 3x^2 + 5x - e^x$ **e** $y = (e^x + 1)^3$ **f** $y = (e^x + 5)^7$

g $y = (2e^x - 3)^2$ **h** $y = xe^x$ **i** $y = \dfrac{e^x}{x}$

j $y = x^2e^x$ **k** $y = e^x(2x + 1)$ **l** $y = \dfrac{e^x}{7x - 3}$

m $y = \dfrac{5x}{e^x}$

2 Find the derivative of:

a $y = e^{2x}$ **b** $y = e^{-x}$ **c** $y = 2e^{3x}$

d $y = -e^{7x}$ **e** $y = -3e^{2x} + x^2$ **f** $y = e^{2x} - e^{-2x}$

g $y = 5e^{-x} - 3x + 2$ **h** $y = xe^{4x}$ **i** $y = \dfrac{2e^{3x} - 3}{x + 1}$

j $y = (9e^{3x} + 2)^5$

3 If $f(x) = x^3 + 3x - e^x$, find $f'(1)$ in terms of e.

4 Find the exact gradient of the tangent to the curve $y = e^x$ at the point $(1, e)$.

5 Find the exact gradient of the normal to the curve $y = e^{2x}$ at the point where $x = 5$.

6 Find the gradient of the tangent to the curve $y = 4e^x$ at the point where $x = 1.6$ correct to 2 decimal places.

7 Find the equation of the tangent to the curve $y = -e^x$ at the point $(1, -e)$.

8 Find the equation of the normal to the curve $y = e^{-x}$ at the point where $x = 3$ in exact form.

9 A population P of insects over time t weeks is given by $P = 3e^{1.4t} + 12\ 569$.

a What is the initial population?

b Find the rate of change in the number of insects after:

i 3 weeks **ii** 7 weeks

10 The displacement of a particle over time t seconds is given by $x = 2e^{4t}$ m.

a What is the initial displacement?

b What is the exact velocity after 10 s?

c Find the acceleration after 2 s correct to 1 decimal place.

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11 The displacement of an object in cm over time t seconds is given by $x = 6e^{-0.34t} - 5$. Find:

a the initial displacement

b the initial velocity

c the displacement after 4 s

d the velocity after 9 s

e the acceleration after 2 s

12 The volume V of a balloon in mm^3 as it expands over time t seconds is given by $V = 3e^{0.8t}$.

a Find the volume of the balloon at:

i 3 s **ii** 5 s

b Find the rate at which the volume is increasing at:

i 3 s **ii** 5 s

13 The population of a city is changing over t years according to the formula $P = 34\ 500e^{0.025t}$.

a Find (to the nearest whole number) the population after:

i 5 years **ii** 10 years

b Find the rate at which the population is changing after:

i 5 years **ii** 10 years

14 The depth of water (in metres) in a dam is decreasing over t months according to the formula $D = 3e^{-0.017t}$.

a Find correct to 2 decimal places the depth after:

i 1 month **ii** 2 months **iii** 3 months

b Find correct to 3 decimal places the rate at which the depth is changing after:

i 1 month **ii** 2 months **iii** 3 months

8.04 Logarithms

The **logarithm** of a positive number, y, is the **power** to which a **base**, a, must be raised in order to produce the number y. For example, $\log_2 8 = 3$ because $2^3 = 8$.

WS Logarithms

If $y = a^x$ then x is called the **logarithm of y to the base a**.

Just as the exponential function $y = a^x$ is defined for positive bases only ($a > 0$), logarithms are also defined for $a > 0$. Furthermore, $a \neq 1$ because $1^x = 1$ for all values of x.

Logarithms

Logarithms

$$\text{If } y = a^x \text{ then } x = \log_a y \qquad (a > 0, a \neq 1, y > 0)$$

Logarithms are related to exponential functions and allow us to solve equations like $2^x = 5$.

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EXAMPLE 11

a Write $\log_4 x = 3$ in index form and solve for x.

b Write $5^2 = 25$ in logarithm form.

c Solve $\log_x 36 = 2$.

d Evaluate $\log_3 81$.

e Find the value of $\log_2 \frac{1}{4}$.

Solution

a $\log_a y = x$ means $y = a^x$

$\log_4 x = 3$ means $x = 4^3$

So $\quad x = 64$

b $y = a^x$ means $\log_a y = x$

So $25 = 5^2$ means $\log_5 25 = 2$

c $\log_x 36 = 2$ means $36 = x^2$

$$\begin{aligned} x &= \sqrt{36} \\ &= 6 \end{aligned}$$

Note: x is the base, so $x > 0$.

d $\log_3 81 = x$ means $81 = 3^x$

Solving $3^x = 81$:

$3^x = 3^4$

So $\quad x = 4$

$\log_3 81 = 4$

e Let $\quad \log_2 \frac{1}{4} = x$

$$\begin{aligned} \text{Then} \quad 2^x &= \frac{1}{4} \\ &= \frac{1}{2^2} \\ &= 2^{-2} \\ \therefore x &= -2 \end{aligned}$$

So $\quad \log_2 \frac{1}{4} = -2$

EXAMPLE 12

Simplify:

a $\log_8 1$ **b** $\log_8 8$ **c** $\log_8 8^3$

d $\log_a a^x$ **e** $3^{\log_3 7}$ **f** $a^{\log_a x}$

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Solution

a $\log_8 1 = 0$ because $8^0 = 1$

b $\log_8 8 = 1$ because $8^1 = 8$

c $\log_8 8^3 = 3$ because $8^3 = 8^3$

d $\log_a a^x = x$ because $a^x = a^x$

e Let $\log_3 7 = y$

Then $3^y = 7$

So substituting for y:

$3^{\log_3 7} = 7$

f Let $\log_a x = y$

Then $a^y = x$

So substituting for y:

$a^{\log_a x} = x$

Notice that logarithms and exponentials are inverse operations.

Properties of logarithms

$$\log_a a = 1$$

$$\log_a 1 = 0$$

$$\log_a a^x = x$$

$$a^{\log_a x} = x$$

Common logarithms and natural logarithms

There are 2 types of logarithms that you can find on your calculator.

- **Common logarithms (base 10)**: $\log_{10} x$ or $\log x$
- **Natural (Naperian) logarithms (base *e*)**: $\log_e x$ or $\ln x$

EXAMPLE 13

a Find $\log_{10} 5.3$ correct to 1 decimal place.

b Evaluate $\log_e 80$ correct to 3 significant figures.

c Loudness in decibels is given by the formula $L = 10 \log_{10}\left(\frac{I}{I_0}\right)$ where I_0 is threshold sound, or sound that can barely be heard. Sound louder than 85 decibels can cause hearing damage.

i The loudness of a vacuum cleaner is 10 000 000 times the threshold level, or $10\ 000\ 000 I_0$. How many decibels is this?

ii If the loudness of the sound of rustling leaves is 20 dB, find its loudness in terms of I_0.

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Solution

a $\log_{10} 5.3 = 0.724\ 2\ \ldots$ log 5.3 =

≈ 0.7

b $\log_e 80 = 4.382\ 0\ldots$ ln 80 =

≈ 4.38

c **i** $L = 10 \log_{10}\left(\dfrac{10\,000\,000 I_0}{I_0}\right)$

$= 10 \log_{10} (10\ 000\ 000)$

$= 10 \times 7$

$= 70$

So the loudness of the vacuum cleaner is 70 dB.

ii $L = 10 \log_{10}\left(\dfrac{I}{I_0}\right)$

$20 = 10 \log_{10}\left(\dfrac{I}{I_0}\right)$

$2 = \log_{10}\left(\dfrac{I}{I_0}\right)$

Using the definition of a logarithm:

$10^2 = \dfrac{I}{I_0}$

$100 = \dfrac{I}{I_0}$

$100 I_0 = I$

So the loudness of rustling leaves is 100 times threshold sound.

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DID YOU KNOW?

The origins of logarithms

John Napier (1550–1617), a Scottish theologian and an amateur mathematician, was the first to invent logarithms. These 'natural', or 'Naperian', logarithms were based on e. Napier originally used the compound interest formula to find the value of e.

Napier was also one of the first mathematicians to use decimals rather than fractions. He invented decimal notation, using either a comma or a point. The point was used in England, but some European countries use a comma.

Henry Briggs (1561–1630), an Englishman who was a professor at Oxford, decided that logarithms would be more useful if they were based on 10 (our decimal system). Briggs painstakingly produced a table of common logarithms correct to 14 decimal places.

The work on logarithms was greatly appreciated by **Kepler**, **Galileo** and other astronomers at the time, since they allowed the computation of very large numbers.

Exercise 8.04 Logarithms

1 Evaluate:

a $\log_2 16$ **b** $\log_4 16$ **c** $\log_5 125$
d $\log_3 3$ **e** $\log_7 49$ **f** $\log_7 7$
g $\log_5 1$ **h** $\log_2 128$ **i** $\log_8 8$

2 Evaluate:

a $2^{\log_2 3}$ **b** $7^{\log_7 4}$ **c** $3^{\log_3 29}$

3 Evaluate:

a $3 \log_2 8$ **b** $\log_5 25 + 1$ **c** $3 - \log_3 81$
d $4 \log_3 27$ **e** $2 \log_{10} 10\,000$ **f** $1 + \log_4 64$
g $3 \log_4 64 + 5$ **h** $\dfrac{\log_3 9}{2}$ **i** $\dfrac{\log_8 64+4}{\log_2 8}$

4 Evaluate:

a $\log_2 \dfrac{1}{2}$ **b** $\log_3 \sqrt{3}$ **c** $\log_4 2$ **d** $\log_5 \dfrac{1}{25}$
e $\log_7 \sqrt[4]{7}$ **f** $\log_3 \dfrac{1}{\sqrt[3]{3}}$ **g** $\log_4 \dfrac{1}{2}$ **h** $\log_8 2$
i $\log_6 6\sqrt{6}$ **j** $\log_2 \dfrac{\sqrt{2}}{4}$

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5 Evaluate correct to 2 decimal places:

a $\log_{10} 1200$ **b** $\log_{10} 875$ **c** $\log_e 25$

d $\ln 140$ **e** $5 \ln 8$ **f** $\log_{10} 350 + 4.5$

g $\dfrac{\log_{10} 15}{2}$ **h** $\ln 9.8 + \log_{10} 17$ **i** $\dfrac{\log_{10} 30}{\log_e 30}$

6 Write in logarithmic form:

a $3^x = y$ **b** $5^x = z$ **c** $x^2 = y$ **d** $2^b = a$

e $b^3 = d$ **f** $y = 8^x$ **g** $y = 6^x$ **h** $y = e^x$

i $y = a^x$ **j** $Q = e^x$

7 Write in index form:

a $\log_3 5 = x$ **b** $\log_a 7 = x$ **c** $\log_3 a = b$

d $\log_x y = 9$ **e** $\log_a b = y$ **f** $y = \log_2 6$

g $y = \log_3 x$ **h** $y = \log_{10} 9$ **i** $y = \ln 4$

8 Solve for x, correct to 1 decimal place where necessary:

a $\log_{10} x = 6$ **b** $\log_3 x = 5$ **c** $\log_x 343 = 3$

d $\log_x 64 = 6$ **e** $\log_5 \dfrac{1}{5} = x$ **f** $\log_x \sqrt{3} = \dfrac{1}{2}$

g $\ln x = 3.8$ **h** $3 \log_{10} x - 2 = 10$ **i** $\log_4 x = \dfrac{3}{2}$

9 Evaluate y given that $\log_y 125 = 3$.

10 If $\log_{10} x = 1.65$, evaluate x correct to 1 decimal place.

11 Evaluate b to 3 significant figures if $\log_e b = 0.894$.

12 Find the value of $\log_2 1$. What is the value of $\log_a 1$?

13 Evaluate $\log_5 5$. What is the value of $\log_a a$?

14 a Evaluate $\ln e$ without a calculator.

b Using a calculator, evaluate:

i $\log_e e^3$ **ii** $\log_e e^2$ **iii** $\ln_e e^5$ **iv** $\log_e \sqrt{e}$

v $\ln_e \dfrac{1}{e}$ **vi** $e^{\ln 2}$ **vii** $e^{\ln 3}$ **viii** $e^{\ln 5}$

ix $e^{\ln 7}$ **x** $e^{\ln 1}$ **xi** $e^{\ln e}$

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15 A class was given musical facts to learn. The students were then tested on these facts and each week they were given similar tests to find out how much they were able to remember. The formula $A = 85 - 55 \log_{10}(t + 2)$ seemed to model the average score after t weeks.

a What was the initial average score?

b What was the average score after:

i 1 week? **ii** 3 weeks?

c After how many weeks was the average score 30?

16 The pH of a solution is defined as $\text{pH} = -\log [H^+]$ where $[H^+]$ is the hydrogen ion concentration. A solution is acidic if its pH is less than 7, alkaline if pH is greater than 7 and neutral if pH is 7. For each question find its pH and state whether it is acidic, alkaline or neutral.

a Fruit juice whose hydrogen ion concentration is 0.0035

b Water with $[H^+] = 10^{-7}$

c Baking soda with $[H^+] = 10^{-9}$

d Coca Cola whose hydrogen ion concentration is 0.01

e Bleach with $[H^+] = 1.2 \times 10^{-12}$

f Coffee with $[H^+] = 0.000\ 01$

17 If $f(x) = \log x$ and $g(x) = 2x - 7$, find:

a $f(g(x))$ **b** $g(f(x))$

INVESTIGATION

HISTORY OF BASES AND NUMBER SYSTEMS

Common logarithms use base 10 like our decimal number system. We might have developed a different system if we had a different number of fingers! The Mayans, in ancient times, used base 20 for their number system since they counted with both their fingers and toes.

1 Research the history and types of other number systems, including those of Aboriginal and Torres Strait Islander peoples. Did any cultures use systems other than base 10? Why?

2 Explore computer-based systems. Computers have used both binary (base 2) and octal (base 8). Find out why these bases are used.

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8.05 Logarithm laws

Logarithm laws

Logarithm laws

Because logarithms are just another way of writing indices (powers), there are logarithm laws that correspond to the index laws.

$$\log_a (xy) = \log_a x + \log_a y$$

Proof

Let $x = a^m$ and $y = a^n$

Then $m = \log_a x$ and $n = \log_a y$

$xy = a^m \times a^n$

$= a^{m+n}$

$\therefore \log_a (xy) = m + n$ (by definition)

$= \log_a x + \log_a y$

$$\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$$

Proof

Let $x = a^m$ and $y = a^n$

Then $m = \log_a x$ and $n = \log_a y$

$\frac{x}{y} = a^m \div a^n$

$= a^{m-n}$

$\therefore \log_a \left(\frac{x}{y}\right) = m - n$ (by definition)

$= \log_a x - \log_a y$

$$\log_a x^n = n \log_a x$$

Proof

Let $x = a^m$

Then $m = \log_a x$

$x^n = (a^m)^n$

$= a^{mn}$

$\therefore \log_a x^n = mn$ (by definition)

$= n \log_a x$

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$$\log_a\left(\frac{1}{x}\right) = -\log_a x$$

Proof

$$\begin{aligned}\log_a\left(\frac{1}{x}\right) &= \log_a 1 - \log_a x\\ &= 0 - \log_a x\\ &= -\log_a x\end{aligned}$$

EXAMPLE 14

Logarithm laws

a Given $\log_5 3 = 0.68$ and $\log_5 4 = 0.86$, find:

i $\log_5 12$ **ii** $\log_5 0.75$ **iii** $\log_5 9$ **iv** $\log_5 20$

b Solve $\log_2 12 = \log_2 3 + \log_2 x$.

c Simplify $\log_a 21$ if $\log_a 3 = p$ and $\log_a 7 = q$.

d The formula for measuring R, the strength of an earthquake on the Richter scale, is $R = \log\left(\frac{I}{S}\right)$ where I is the maximum seismograph signal of the earthquake being measured and S is the signal of a standard earthquake.

Show that:

i $\log I = R + \log S$ **ii** $I = S(10^R)$

Solution

a i
$$\begin{aligned}\log_5 12 &= \log_5 (3 \times 4)\\ &= \log_5 3 + \log_5 4\\ &= 0.68 + 0.86\\ &= 1.54\end{aligned}$$

ii
$$\begin{aligned}\log_5 0.75 &= \log_5 \frac{3}{4}\\ &= \log_5 3 - \log_5 4\\ &= 0.68 - 0.86\\ &= -0.18\end{aligned}$$

iii
$$\begin{aligned}\log_5 9 &= \log_5 3^2\\ &= 2\log_5 3\\ &= 2 \times 0.68\\ &= 1.36\end{aligned}$$

iv
$$\begin{aligned}\log_5 20 &= \log_5 (5 \times 4)\\ &= \log_5 5 + \log_5 4\\ &= 1 + 0.86\\ &= 1.86\end{aligned}$$

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b $\log_2 12 = \log_2 3 + \log_2 x$

$= \log_2 3x$

So $12 = 3x$

$4 = x$

c $\log_a 21 = \log_a (3 \times 7)$

$= \log_a 3 + \log_a 7$

$= p + q$

d **i** $R = \log\left(\frac{I}{S}\right)$

$= \log I - \log S$

$R + \log S = \log I$

ii $R = \log\left(\frac{I}{S}\right)$

$\frac{I}{S} = 10^R$

$I = S(10^R)$

Change of base

If we need to evaluate logarithms such as $\log_5 2$, we use the change of base formula.

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Proof

Let $y = \log_a x$

Then $x = a^y$

Take logarithms to the base b of both sides of the equation:

$\log_b x = \log_b a^y$

$= y \log_b a$

$\therefore \frac{\log_b x}{\log_b a} = y$

$= \log_a x$

To find the logarithm of any number, such as $\log_5 2$, you can change it to either $\log_{10} x$ or $\log_e x$.

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EXAMPLE 15

a Evaluate $\log_5 2$ correct to 2 decimal places.

b Find the value of $\log_2 3$ to 1 decimal place.

Solution

a $\log_5 2 = \dfrac{\log 2}{\log 5}$

≈ 0.43

b $\log_2 3 = \dfrac{\log 3}{\log 2}$

≈ 1.6

Exercise 8.05 Logarithm laws

1 Simplify:

a $\log_a 4 + \log_a y$ **b** $\log_a 4 + \log_a 5$
c $\log_a 12 - \log_a 3$ **d** $\log_a b - \log_a 5$
e $3 \log_x y + \log_x z$ **f** $2 \log_k 3 + 3 \log_k y$
g $5 \log_a x - 2 \log_a y$ **h** $\log_a x + \log_a y - \log_a z$
i $\log_{10} a + 4 \log_{10} b + 3 \log_{10} c$ **j** $3 \log_3 p + \log_3 q - 2 \log_3 r$
k $\log_4 \dfrac{1}{n}$ **l** $\log_x \dfrac{1}{6}$

2 Evaluate:

a $\log_5 5^2$ **b** $\log_7 7^6$

3 Given $\log_7 2 = 0.36$ and $\log_7 5 = 0.83$, find:

a $\log_7 10$ **b** $\log_7 0.4$ **c** $\log_7 20$
d $\log_7 25$ **e** $\log_7 8$ **f** $\log_7 14$
g $\log_7 50$ **h** $\log_7 35$ **i** $\log_7 98$

4 Use the logarithm laws to evaluate:

a $\log_5 50 - \log_5 2$ **b** $\log_2 16 + \log_2 4$
c $\log_4 2 + \log_4 8$ **d** $\log_5 500 - \log_5 4$
e $\log_9 117 - \log_9 13$ **f** $\log_8 32 + \log_8 16$
g $3 \log_2 2 + 2 \log_2 4$ **h** $2 \log_4 6 - (2 \log_4 3 + \log_4 2)$
i $\log_6 4 - 2 \log_6 12$ **j** $2 \log_3 6 + \log_3 18 - 3 \log_3 2$

5 If $\log_a 3 = x$ and $\log_a 5 = y$, find an expression in terms of x and y for:

a $\log_a 15$ **b** $\log_a 0.6$ **c** $\log_a 27$

d $\log_a 25$ **e** $\log_a 9$ **f** $\log_a 75$

g $\log_a 3a$ **h** $\log_a \frac{a}{5}$ **i** $\log_a 9a$

6 If $\log_a x = p$ and $\log_a y = q$, find, in terms of p and q:

a $\log_a xy$ **b** $\log_a y^3$ **c** $\log_a \frac{y}{x}$ **d** $\log_a x^2$

e $\log_a xy^5$ **f** $\log_a \frac{x^2}{y}$ **g** $\log_a ax$ **h** $\log_a \frac{a}{y^2}$

i $\log_a a^3y$ **j** $\log_a \frac{x}{ay}$

7 If $\log_a b = 3.4$ and $\log_a c = 4.7$, evaluate:

a $\log_a \frac{c}{b}$ **b** $\log_a bc^2$ **c** $\log_a (bc)^2$

d $\log_a abc$ **e** $\log_a a^2c$ **f** $\log_a b^7$

g $\log_a \frac{a}{c}$ **h** $\log_a a^3$ **i** $\log_a bc^4$

8 Solve:

a $\log_4 12 = \log_4 x + \log_4 3$ **b** $\log_3 4 = \log_3 y - \log_3 7$

c $\log_a 6 = \log_a x - 3\log_a 2$ **d** $\log_2 81 = 4\log_2 x$

e $\log_x 54 = \log_x k + 2\log_x 3$

9 **a** Change the subject of $\text{dB} = 10\log\left(\frac{I}{I_0}\right)$ to I.

b Find the value of I in terms of I_0 when dB = 45.

10 **a** Show that the formula $A = 100 - 50\log(t + 1)$ can be written as:

i $\log(t+1) = \frac{100 - A}{50}$ **ii** $t = 10^{\frac{100-A}{50}} - 1$

b Hence find:

i A when $t = 3$ **ii** t when $A = 75$

11 Evaluate to 2 decimal places:

a $\log_4 9$ **b** $\log_6 25$ **c** $\log_9 200$ **d** $\log_2 12$

e $\log_3 23$ **f** $\log_8 250$ **g** $\log_5 9.5$ **h** $2\log_4 23.4$

i $7 - \log_7 108$ **j** $3\log_{11} 340$

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8.06 Logarithmic functions

Exponential and log graphs match-up

A **logarithmic function** is a function of the form $y = \log_a x$.

EXAMPLE 16

Sketch the graph of $y = \log_2 x$.

Solution

y-intercept ($x = 0$): No y-intercept because $x > 0$.

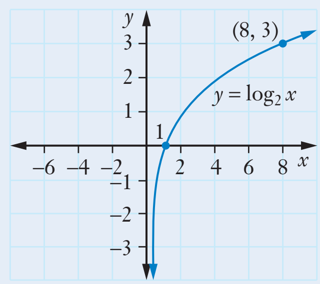
x-intercept ($y = 0$): $0 = \log_2 x$

$x = 2^0 = 1$, so x-intercept is 1 ($y = 0$).

Complete a table of values.

$y = \log_2 x$ means $x = 2^y$. For $x = 6$ in the table, use the change of base formula, $\log_2 x = \dfrac{\log x}{\log 2}$.

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	6	8
y	–2	–1	0	1	2	2.58	3



Logarithmic functions

- The logarithmic function $y = \log_a x$ is the inverse function of an exponential function $y = a^x$.
- Domain $(0, \infty)$, range $(-\infty, \infty)$.
- $x > 0$ so the curve is always to the right of the y-axis (no y-intercept).
- The y-axis is an **asymptote**.
- The x-intercept is always 1 because $\log_a 1 = 0$.

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EXAMPLE 17

Sketch the graph of:

a $y = \log_e x - 1$ **b** $y = 3\log_{10} x + 4$

Solution

a No y-intercept ($x = 0$) because $\log_e 0$ is undefined. The y-axis is an asymptote.

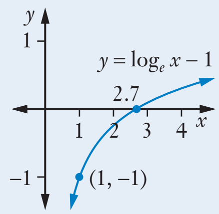
x-intercept ($y = 0$)

$0 = \log_e x - 1$

$1 = \log_e x$

$x = e^1$

≈ 2.7



Notice that this is the graph of $y = \log_e x$ moved down 1 unit.

Complete a table of values for this graph using the ln key on the calculator.

x	1	2	3	4
y	−1	−0.3	0.1	0.4

b Complete a table of values using the log key on the calculator.

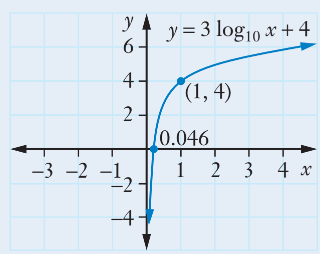
$y = 3\log_{10} x + 4$

x	1	2	3	4
y	4	4.9	5.4	5.8

No y-intercept.

For x-intercept, $y = 0$:

$$0 = 3\log_{10} x + 4$$
$$-4 = 3\log_{10} x$$
$$-\frac{4}{3} = \log_{10} x$$
$$10^{-\frac{4}{3}} = x$$
$$x = 0.04641\ldots$$
$$\approx 0.046$$



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EXAMPLE 18

a Sketch the graphs of $y = e^x$, $y = \log_e x$ and $y = x$ on the same set of axes.

b What relationship do these graphs have?

c If $f(x) = \log_a x$, sketch the graph of $y = -f(x)$ and state its domain and range.

Solution

a Drawing $y = e^x$ gives an exponential curve with y-intercept 1.

Find another point, say $x = 2$:

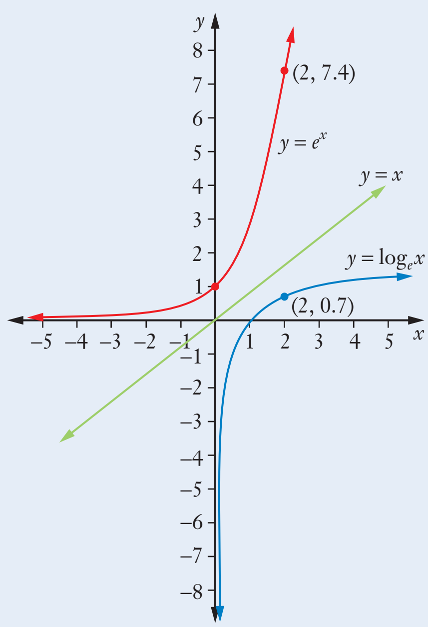
$$
\begin{aligned}
y &= e^2 \\
&= 7.3890\ldots \\
&\approx 7.4
\end{aligned}
$$

Drawing $y = \log_e x$ gives a logarithmic curve with x-intercept 1.

Find another point, say $x = 2$:

$$
\begin{aligned}
y &= \ln 2 \\
&= 0.6931\ldots \\
&\approx 0.7
\end{aligned}
$$

$y = x$ is a linear function with gradient 1 and y-intercept 0.



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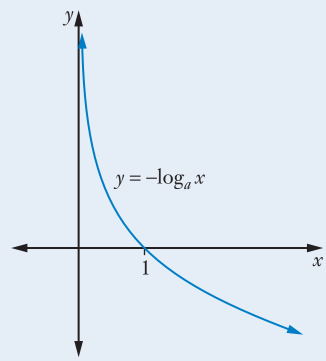
b The graphs of $y = e^x$ and $y = \log_e x$ are reflections of each other in the line $y = x$. They are **inverse functions**.

c Given $f(x) = \log_a x$,

$y = -f(x)$

$= -\log_a x$

This is a reflection of $f(x)$ in the x-axis.



Domain $(0, \infty)$, range $(-\infty, \infty)$

The exponential and logarithmic functions

$f(x) = a^x$ and $f(x) = \log_a x$ are inverse functions. Their graphs are reflections of each other in the line $y = x$.

INVESTIGATION

GRAPHS OF LOGARITHMIC FUNCTIONS

1 Substitute different values of x into the logarithmic function $y = \log x$: positive, negative and zero. What do you notice?

2 Use a graphics calculator or graphing software to sketch the graphs of different logarithmic functions such as

a $y = \log_2 x, y = \log_3 x, y = \log_4 x, y = \log_5 x, y = \log_6 x$

b $y = \log_2 x + 1, y = \log_2 x + 2, y = \log_2 x + 3, y = \log_2 x - 1, y = \log_2 x - 2$

c $y = 2\log_2 x, y = 3\log_2 x, y = -\log_2 x, y = -2\log_2 x, y = -3\log_2 x$

d $y = 2\log_2 x + 1, y = 2\log_2 x + 2, y = 2\log_2 x + 3, y = 2\log_2 x - 1, y = 2\log_2 x - 2$

e $y = 3\log_4 x + 1, y = 5\log_3 x + 2, y = -\log_5 x + 3, y = -2\log_2 x - 1, y = 4\log_7 x - 2$

3 Try sketching the graph of $y = \log_{-2} x$. What does the table of values look like? Are there any discontinuities on the graph? Why? Could you find the domain and range? Use a graphics calculator or graphing software to sketch this graph. What do you find?

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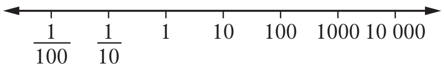
Logarithmic scales

It is difficult to describe and graph exponential functions because their y values increase so quickly. We use logarithms and **logarithmic scales** to solve this problem.

On a base 10 logarithmic scale, an axis or number line has units that don't increase by 1, but by powers of 10.



Examples of base 10 logarithmic scales are:

- the Richter scale for measuring earthquake magnitude
- the pH scale for measuring acidity in chemistry
- the decibel scale for measuring loudness
- the octave (frequency) scale in music

EXAMPLE 19

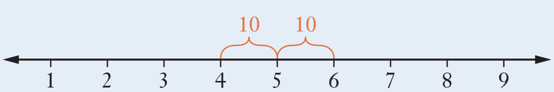
a Ged finds that the pH of soil is 4 in the eastern area of his garden and 6 in the western area. The pH formula is logarithmic and pH < 7 is acidic. What is the difference in acidity in these 2 areas of the garden?

b If Ged finds another area with a pH of 3.6, how much more acidic is this area than the eastern area?

Solution

a The difference in pH between 4 and 6 is 2. But this is a logarithmic scale.

Each interval on a logarithmic scale is a multiple of 10.



So the difference is $10 \times 10 = 10^2 = 100$.

The lower the pH, the more acidic. So the soil in the eastern area is 100 times more acidic than the soil in the western area.

b The difference in pH between 4 and 3.6 is 0.4.

So the difference is $10^{0.4} = 2.5118 \ldots \approx 2.5$.

The soil in this area is about 2.5 times more acidic than the soil in the eastern area.

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Exercise 8.06 Logarithmic functions

1 Sketch the graph of each logarithmic function and state its domain and range.

a $y = \log_3 x$ **b** $f(x) = 2\log_4 x$ **c** $y = \log_2 x + 1$

d $y = \log_5 x - 1$ **e** $f(x) = \log_4 x - 2$ **f** $y = 5\ln x + 3$

g $f(x) = -3\log_{10} x + 2$

2 Sketch the graphs of $y = 10^x$, $y = \log_{10} x$ and $y = x$ on the same number plane. What do you notice about the relationship of the curves to the line?

3 Sketch the graph of $f(x) = \log_2 x$ and $y = \log_2 (-x)$ on the same set of axes and describe their relationship.

4 **a** Sketch the graphs of $y = \log_2 x$, $y = 2^x$ and $y = x$ on the same set of axes.

b Find the inverse function of $y = \log_2 x$.

5 This table lists some of the earthquakes experienced in Australia.

Year	Location	Strength on Richter scale
1989	Newcastle NSW	5.6
1997	Collier Bay WA	6.3
2001	Swan Hill Vic	4.8
2010	Kalgoorlie WA	5.2
2015	Coral Sea Qld	5.5
2017	Orange NSW	4.3
2018	Coffs Harbour NSW	4.2

The Richter scale for earthquakes is logarithmic. Use the table to find the difference in magnitude (correct to the nearest whole number) between the earthquakes in:

a Newcastle and Swan Hill

b Collier Bay and Orange

c Newcastle and Orange

d Coral Sea and Kalgoorlie

e Collier Bay and Coffs Harbour

6 The decibel (dB) scale for loudness is logarithmic. Find (correct to the nearest whole number) the difference in loudness between:

a 20 and 23 dB **b** 40 and 41 dB **c** 65.2 and 66.5 dB

d 85.4 and 88.9 dB **e** 52.3 and 58.6 dB

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8.07 Exponential equations

Exponential equations can be solved using logarithms or the change of base formula.

Exponential equations

Logarithmic and exponential equations

Solving exponential equations

Using exponential models

EXAMPLE 20

Solve $5^x = 7$ correct to 1 decimal place.

Solution

Method 1: Logarithms

Take logarithms of both sides:

$\log 5^x = \log 7$

$x \log 5 = \log 7$

$$x = \frac{\log 7}{\log 5}$$

$= 1.2090\ldots$

≈ 1.2

Method 2: Change of base formula

Convert to logarithm form:

$5^x = 7$ means $\log_5 7 = x$

Using the change of base to evaluate x:

$x = \log_5 7$

$$= \frac{\log 7}{\log 5}$$

$= 1.2090\ldots$

≈ 1.2

EXAMPLE 21

a Solve $e^{3.4x} = 100$ correct to 2 decimal places.

b The temperature T in °C of a metal as it cools down over t minutes is given by $T = 27 + 219e^{-0.032t}$. Find, correct to 1 decimal place, the time it takes to cool down to 100°C.

Exponential functions

Solution

a With an equation involving e we use $\ln x$, which is $\log_e x$.

Take natural logs of both sides:

$\ln e^{3.4x} = \ln 100$

$3.4x = \ln 100$ $\quad$ $\ln x$ and e^x are inverses

$$x = \frac{\ln 100}{3.4}$$

$= 1.3544\ldots$

≈ 1.35

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b When $T = 100$:

$$100 = 27 + 219e^{-0.032t}$$
$$73 = 219e^{-0.032t}$$
$$\frac{73}{219} = e^{-0.032t}$$
$$\log_e\left(\frac{73}{219}\right) = \log_e(e^{-0.032t})$$
$$= -0.032t$$

$$t = \frac{\log_e\left(\frac{73}{219}\right)}{-0.032}$$
$$= 34.3316\ldots$$
$$= 34.3 \text{ to 1 d.p.}$$

So it takes 34.3 minutes to cool down to 100°C.

Exercise 8.07 Exponential equations

1 Solve each equation correct to 2 significant figures:

a $4^x = 9$ **b** $3^x = 5$ **c** $7^x = 14$ **d** $2^x = 15$
e $5^x = 34$ **f** $6^x = 60$ **g** $2^x = 76$ **h** $4^x = 50$
i $3^x = 23$ **j** $9^x = 210$

2 Solve, correct to 2 decimal places:

a $2^x = 6$ **b** $5^y = 15$ **c** $3^x = 20$ **d** $7^m = 32$
e $4^k = 50$ **f** $3^t = 4$ **g** $8^x = 11$ **h** $2^p = 57$
i $4^x = 81.3$ **j** $6^n = 102.6$

3 Solve, to 1 decimal place:

a $3^{x+1} = 8$ **b** $5^{3n} = 71$ **c** $2^{x-3} = 12$
d $4^{2n-1} = 7$ **e** $7^{5x+2} = 11$ **f** $8^{3-n} = 5.7$
g $2^{x+2} = 18.3$ **h** $3^{7k-3} = 32.9$ **i** $9^{\frac{x}{2}} = 50$

4 Solve each equation correct to 3 significant figures:

a $e^x = 200$ **b** $e^{3t} = 5$ **c** $2e^t = 75$
d $45 = e^x$ **e** $3000 = 100e^n$ **f** $100 = 20e^{3t}$
g $2000 = 50e^{0.15t}$ **h** $15\,000 = 2000e^{0.03k}$ **i** $3Q = Qe^{0.02t}$

5 The amount A of money in a bank account after n years grows with compound interest according to the formula $A = 850(1.025)^n$.

a Find:

i the initial amount in the bank **ii** the amount after 7 years.

b Find how many years it will take for the amount in the bank to be \$1000.

6 The population of a city is given by $P = 35\,000e^{0.024t}$ where t is time in years.

a Find the population:

i initially **ii** after 10 years **iii** after 50 years.

b Find when the population will reach:

i 80 000 **ii** 200 000

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7 A species of wattle is gradually dying out in a Blue Mountains region. The number of wattle trees over time t years is given by $N = 8900e^{-0.048t}$.

Photo courtesy Margaret Grove

a Find the number of wattle trees:

i initially

ii after 5 years

iii after 70 years.

b After how many years will there be:

i 5000 wattle trees?

ii 200 wattle trees?

8 A formula for the mass M g of plutonium after t years is given by $M = 100e^{-0.000\,03t}$. Find:

a initial mass **b** mass after 50 years **c** mass after 500 years

d its half-life (the time taken to decay to half of its initial mass)

9 The temperature of an electronic sensor is given by the formula $T = 18 + 12e^{0.002t}$ where t is in hours.

a What is the temperature of the sensor after 5 hours?

b When the temperature reaches 50°C the sensor needs to be shut down to cool. After how many hours does this happen?

10 A particle is moving along a straight line with displacement x cm over time t s according to the formula $x = 5e^t + 23$.

a Find:

i the initial displacement

ii the exact velocity after 20 s

iii the displacement after 6 s

iv the time when displacement is 85 cm

v the time when the velocity is 1000 cm s^{-1}.

b Show that acceleration $a = x - 23$.

c Find the acceleration when displacement is 85 cm.

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8. TEST YOURSELF

Practice quiz

For Questions **1** to **3**, select the correct answer **A**, **B**, **C** or **D**.

1 Simplify $\log_a 15 - \log_a 3$:

A $\log_a 45$ **B** $\dfrac{\log_a 15}{\log_a 3}$ **C** $\log_a 15 \times \log_a 3$ **D** $\log_a 5$

2 Write $a^x = y$ as a logarithm.

A $\log_y x = a$ **B** $\log_a y = x$ **C** $\log_a x = y$ **D** $\log_x a = y$

3 Solve $5^x = 4$ (there is more than one answer).

A $x = \dfrac{\log 4}{\log 5}$ **B** $x = \dfrac{\log 5}{\log 4}$ **C** $x = \dfrac{\ln 4}{\ln 5}$ **D** $x = \dfrac{\ln 5}{\ln 4}$

4 Evaluate:

a $\log_2 8$ **b** $\log_7 7$ **c** $\log_{10} 1000$ **d** $\log_9 81$

e $\log_e e$ **f** $\log_4 64$ **g** $\log_9 3$ **h** $\log_2 \dfrac{1}{2}$

i $\log_5 \dfrac{1}{25}$ **j** $\ln e^3$

5 Evaluate to 3 significant figures:

a $e^2 - 1$ **b** $\log_{10} 95$ **c** $\log_e 26$ **d** $\log_4 7$

e $\log_4 3$ **f** $\ln 50$ **g** $e + 3$ **h** $\dfrac{5e^3}{\ln 4}$

6 Evaluate:

a $e^{\ln 6}$ **b** $e^{\ln 2}$

7 Write in index form:

a $\log_3 a = x$ **b** $\ln b = y$ **c** $\log c = z$

8 If $\log_7 2 = 0.36$ and $\log_7 3 = 0.56$, find the value of:

a $\log_7 6$ **b** $\log_7 8$ **c** $\log_7 1.5$

d $\log_7 14$ **e** $\log_7 3.5$

9 Solve:

a $3^x = 8$ **b** $2^{3x-4} = 3$ **c** $\log_x 81 = 4$ **d** $\log_6 x = 2$

10 Solve $12 = 10e^{0.01t}$.

11 Evaluate $\log_9 8$ to 1 decimal place.

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12 Simplify:

a $5\log_a x + 3\log_a y$ **b** $2\log_x k - \log_x 3 + \log_x p$

13 Evaluate to 2 significant figures:

a $\log_{10} 4.5$ **b** $\ln 3.7$

14 Sketch the graph of $y = 2^x + 1$ and state its domain and range.

15 Solve:

a $2^x = 9$ **b** $3^x = 7$ **c** $5^{x+1} = 6$ **d** $4^{2y} = 11$

e $8^{3n-2} = 5$ **f** $\log_x 16 = 4$ **g** $\log_3 y = 3$ **h** $\log_7 n = 2$

i $\log_x 64 = \frac{1}{2}$ **j** $\log_8 m = \frac{1}{3}$

16 Write as a logarithm:

a $2^x = y$ **b** $5^a = b$ **c** $10^x = y$

d $e^x = z$ **e** $3^{x+1} = y$

17 Sketch the graph of:

a $y = 5(3^{x+2})$ **b** $y = 2(3^x) - 5$ **c** $f(x) = -3^x$ **d** $y = 3(2^{-x})$

18 Sketch the graph of:

a $f(x) = \log_3 x$ **b** $y = 3\ln x - 4$

19 If $\log_x 2 = a$ and $\log_x 3 = b$ find in terms of a and b:

a $\log_x 6$ **b** $\log_x 1.5$ **c** $\log_x 8$

d $\log_x 18$ **e** $\log_x 27$

20 The formula for loudness is $L = 10\log\left(\frac{I}{I_0}\right)$ where I_0 is threshold sound and L is measured in decibels (dB). Find:

a the dB level of a $5500I_0$ sound

b the sound in terms of I_0 if its dB level is 32.

21 Simplify:

a $\log_a \frac{1}{x}$ **b** $\log_e \frac{1}{y}$

22 Evaluate:

a $\log_6 12 + \log_6 3$ **b** $\log 25 + \log 4$ **c** $2\log_4 8$

d $\log_8 72 - \log_8 9$ **e** $\log 53\,000 - \log 53$

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23 Solve correct to 1 decimal place:

a $e^x = 15$ **b** $2.7^x = 21.8$ **c** $10^x = 128.7$

24 The amount of money in the bank after n years is given by $A = 5280(1.019)^n$.

a Find the amount in the bank:

i initially **ii** after 3 years **iii** after 4 years.

b Find how long it will take for the amount of money in the bank to reach:

i \$6000 **ii** \$10 000

25 Differentiate each function.

a $y = e^{3x}$ **b** $y = e^{-2x}$ **c** $y = 5e^{4x}$

d $y = -2e^{8x} + 5x^3 - 1$ **e** $y = x^2e^{2x}$ **f** $y = (4e^{3x} - 1)^9$

g $y = \frac{x}{e^{2x}}$

26 The formula for the number of wombats in a region of New South Wales after t years is $N = 1118 - 37e^{0.032t}$.

a Find the initial number of wombats in this region.

b How many wombats are there after 5 years?

c How long will it take until the number of wombats in the region is:

i 500? **ii** 100?

27 Differentiate:

a $y = e^x + x$ **b** $y = -4e^x$ **c** $y = 3e^{-x}$

d $y = (3 + e^x)^9$ **e** $y = 3x^5e^x$ **f** $y = \frac{e^x}{7x - 2}$

28 An earthquake has magnitude 6.7 and its aftershock has magnitude 4.7 on the base 10 logarithmic Richter scale. How much larger is the first earthquake?

29 Shampoo A has pH 7.2 and shampoo B has pH 8.5. The pH scale is base 10 logarithmic. How much more alkaline is shampoo B?

30 If $f(x) = \log_e x$, $g(x) = e^x$ and $h(x) = 6x^2 - 1$, find:

a $f(h(x))$ **b** $g(h(x))$ **c** $h(g(x))$

d $f(g(x))$ **e** $g(f(x))$

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8. CHALLENGE EXERCISE

1 If $\log_b 2 = 0.6$ and $\log_b 3 = 1.1$, find:

a $\log_b 6b$ **b** $\log_b 8b$ **c** $\log_b 1.5b^2$

2 Find the point of intersection of the curves $y = \log_e x$ and $y = \log_{10} x$.

3 Sketch the graph of $y = \log_2 (x - 1)$ and state its domain and range.

4 By substituting $u = 3^x$, solve $3^{2x} - 3^x - 2 = 0$ correct to 2 decimal places.

5 The pH of a solution is given by $\text{pH} = -\log [\text{H}^+]$ where $[\text{H}^+]$ is the hydrogen ion concentration.

a Show that pH could be given by $\text{pH} = \log \dfrac{1}{[\text{H}^+]}$.

b Show that $[\text{H}^+] = \dfrac{1}{10^{\text{pH}}}$.

c Find the hydrogen ion concentration, to 1 significant figure, of a substance with a pH of:

i 6.3

ii 7.7

6 If $y = 8 + \log_2 (x + 2)$:

a show that $x = 2(2^{y-9} - 1)$

b find, correct to 2 decimal places:

i y when $x = 5$

ii x when $y = 1$

7 Find the equation of **a** the tangent and **b** the normal to the curve $y = 3e^x - 5$ at the point $(2, 3e^2 - 5)$.

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