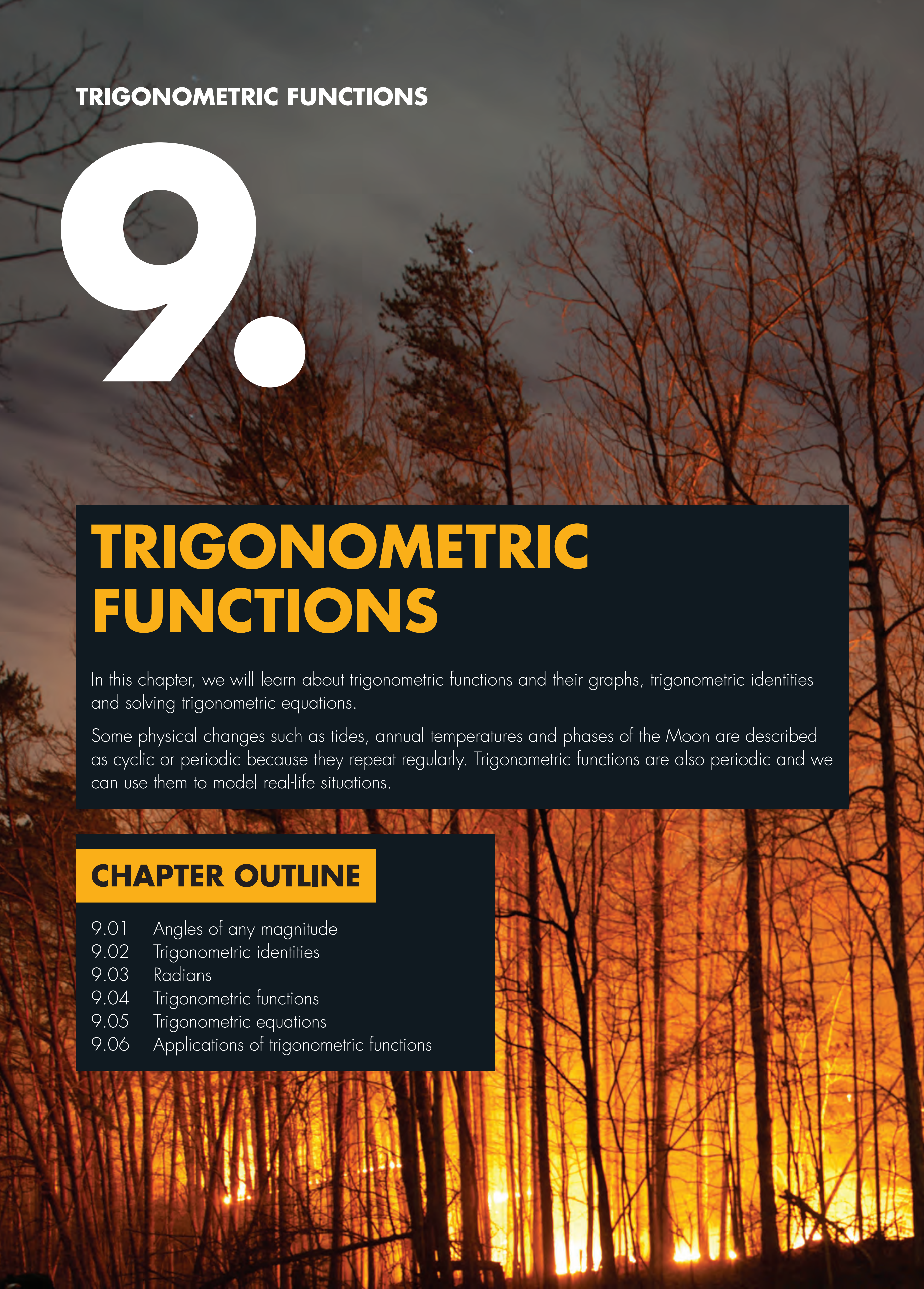


9. TRIGONOMETRIC FUNCTIONS

In this chapter, we will learn about trigonometric functions and their graphs, trigonometric identities and solving trigonometric equations.

Some physical changes such as tides, annual temperatures and phases of the Moon are described as cyclic or periodic because they repeat regularly. Trigonometric functions are also periodic and we can use them to model real-life situations.

CHAPTER OUTLINE

IN THIS CHAPTER YOU WILL:

- evaluate trigonometric ratios for angles of any magnitude in degrees and radians
- use reciprocal trigonometric ratios and trigonometric identities
- solve trigonometric equations
- understand trigonometric functions and sketch their graphs
- examine practical applications of trigonometric functions

TERMINOLOGY

amplitude: The height from the centre of a periodic function to the maximum or minimum values (peaks and troughs of its graph respectively) For $y = k \sin ax$ the amplitude is k

centre: The mean value of a periodic function that is equidistant from the maximum and minimum values. For $y = k \sin ax + c$ the centre is c

identity: An equation that shows the equivalence of 2 algebraic expressions for all values of the variables

period: The length of one cycle of a periodic function on the x-axis, before the function repeats itself. For $y = k \sin ax$ the period is $\frac{2\pi}{a}$

periodic function: A function that repeats itself regularly

phase: A horizontal shift (translation). For $y = k \sin [a(x + b)]$, the phase is b, that is, the graph of $y = k \sin ax$ shifted b units to the left

reciprocal trigonometric ratios: The cosecant, secant and cotangent ratios, which are the reciprocals of sine, cosine and tangent respectively

9.01 Angles of any magnitude

WS

Angles of any magnitude

In Chapter 4, *Trigonometry*, we examined acute and obtuse obtuse angles by looking at angles turning around a unit circle. We can find angles of *any* size by continuing around the circle.

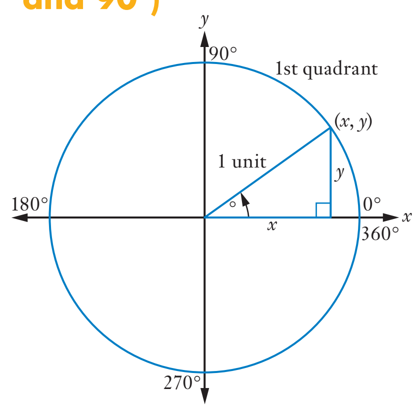
1st quadrant: acute angles (between 0° and 90°)

You can see from the triangle in the unit circle with angle θ that:

$$\sin \theta = y$$

$$\cos \theta = x$$

$$\tan \theta = \frac{y}{x}$$



In the 1st quadrant, x and y are both positive so all ratios are positive in the 1st quadrant.

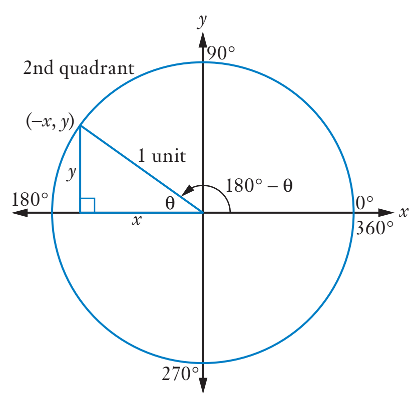
2nd quadrant: obtuse angles (between 90° and 180°)

$\sin \theta = y$ (positive)

$\cos \theta = -x$ (negative)

$\tan \theta = \frac{y}{-x}$ (negative)

The angle that gives θ in the triangle is $180° - \theta$.



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2nd quadrant

$$\sin(180° - \theta) = \sin\theta$$

$$\cos(180° - \theta) = -\cos\theta$$

$$\tan(180° - \theta) = -\tan\theta$$

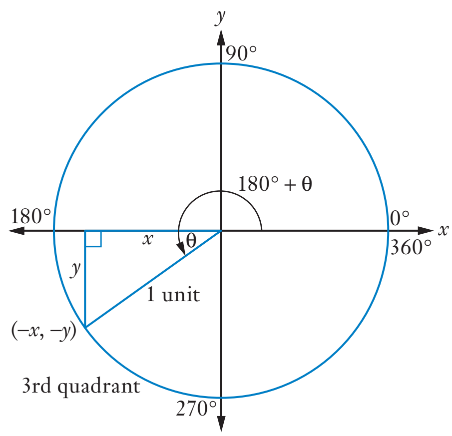
3rd quadrant: angles between 180° and 270°

$\sin\theta = -y$ (negative)

$\cos\theta = -x$ (negative)

$\tan\theta = \frac{-y}{-x} = \frac{y}{x}$ (positive)

The angle that gives θ in the triangle is $180° + \theta$.



3rd quadrant

$$\sin(180° + \theta) = -\sin\theta$$

$$\cos(180° + \theta) = -\cos\theta$$

$$\tan(180° + \theta) = \tan\theta$$

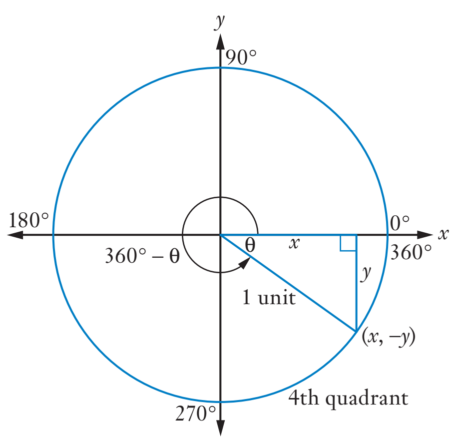
4th quadrant: angles between 270° and 360°

$\sin\theta = -y$ (negative)

$\cos\theta = x$ (positive)

$\tan\theta = \frac{-y}{x}$ (negative)

The angle that gives θ in the triangle is $360° - \theta$.



4th quadrant

$$\sin(360° - \theta) = -\sin\theta$$

$$\cos(360° - \theta) = \cos\theta$$

$$\tan(360° - \theta) = -\tan\theta$$

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Putting all of these results together gives a rule for all 4 quadrants that we usually call the **ASTC rule.**

ASTC rule

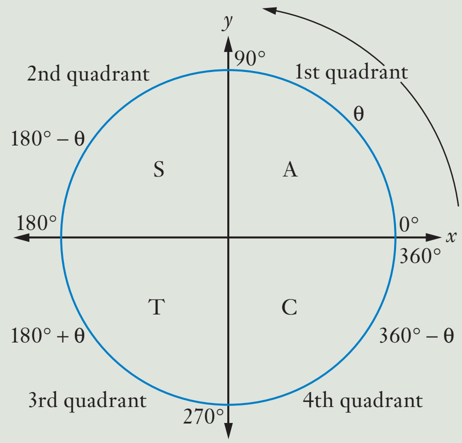
A: All ratios are positive in the 1st quadrant.

S: Sin is positive in the 2nd quadrant (cos and tan are negative).

T: Tan is positive in the 3rd quadrant (sin and cos are negative).

C: Cos is positive in the 4th quadrant (sin and tan are negative).

ASTC can be remembered using the phrase '**A**ll **S**tations **T**o **C**entral'.



EXAMPLE 1

a Find all quadrants where:

i $\sin\theta > 0$ **ii** $\cos\theta < 0$ **iii** $\tan\theta < 0$ and $\cos\theta > 0$

b Find the exact value of:

i $\tan 330°$ **ii** $\sin 225°$

c Simplify $\cos(180° + x)$.

d If $\sin x = -\frac{3}{5}$ and $\cos x > 0$, find the value of $\tan x$.

Solution

a **i** Using the ASTC rule, $\sin\theta > 0$ in the 1st and 2nd quadrants.

ii $\cos\theta > 0$ in the 1st and 4th quadrants, so $\cos\theta < 0$ in the 2nd and 3rd quadrants.

iii $\tan\theta > 0$ in the 1st and 3rd quadrants so $\tan\theta < 0$ in the 2nd and 4th quadrants. Also $\cos\theta > 0$ in the 1st and 4th quadrants.

So $\tan\theta < 0$ and $\cos\theta > 0$ in the 4th quadrant.

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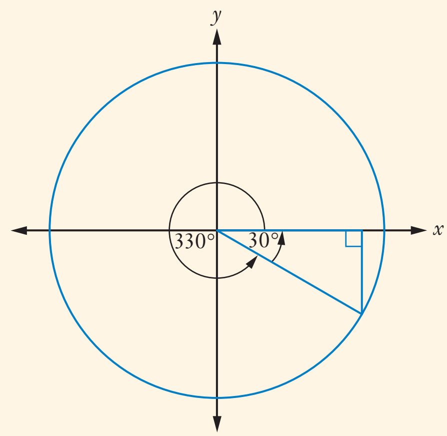
b **i** 330° lies in the 4th quadrant.

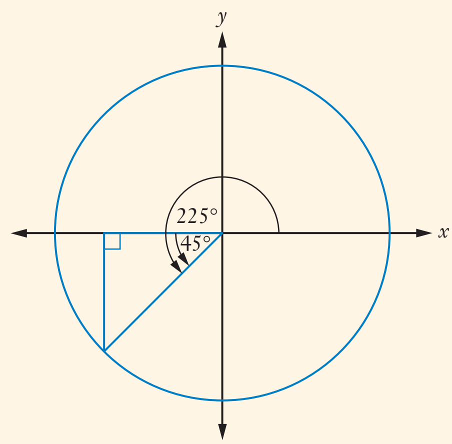
The angle inside the triangle in the 4th quadrant is $360° - 330° = 30°$ and tan is negative in the 4th quadrant.

$$\tan 330° = -\tan 30°$$
$$= -\frac{1}{\sqrt{3}}$$



ii 225° is in the 3rd quadrant. The angle in the triangle in the 3rd quadrant is $225° - 180° = 45°$ and sin is negative in the 3rd quadrant.

$$\sin 225° = -\sin 45°$$
$$= -\frac{1}{\sqrt{2}}$$



c $180° + x$ is in the 3rd quadrant where $\cos x$ is negative.

So $\cos(180° + x) = -\cos x$.

d $\sin x < 0$ and $\cos x > 0$ so x is in the 4th quadrant.

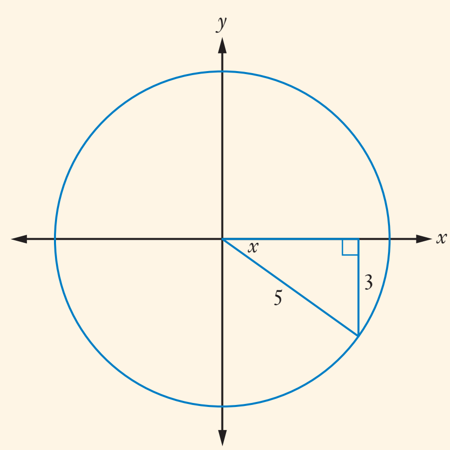
$\sin x = -\frac{3}{5}$

So the opposite side is 3 and the hypotenuse is 5.

By Pythagoras' theorem, the adjacent side is 4. (3, 4, 5 triangle).

$\tan x < 0$ in the 4th quadrant.

So $\tan x = -\frac{3}{4}$.



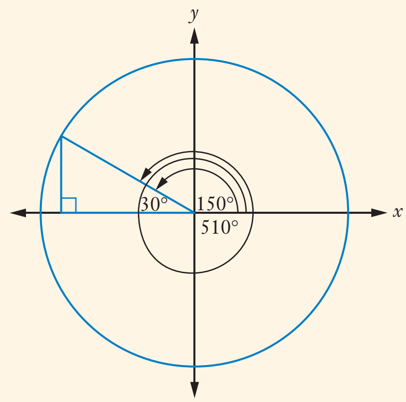
We can find trigonometric ratios of angles greater than 360° by turning around the circle more than once.

EXAMPLE 2

Find the exact value of cos 510°.

Solution

To find cos 510°, we move around the circle more than once.



$\cos(510° - 360°) = \cos(150°)$

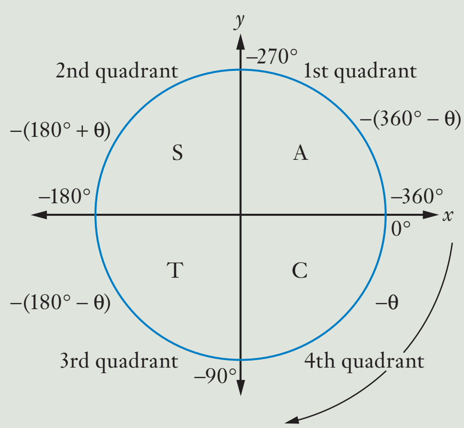
The angle is in the 2nd quadrant where cos is negative. The angle inside the triangle is $180° - 150° = 30°$.

$$\begin{aligned}\text{So } \cos 510° &= \cos 150° \\ &= -\cos 30° \\ &= -\frac{\sqrt{3}}{2}.\end{aligned}$$

Negative angles

The ASTC rule also works for negative angles. These are measured in the opposite direction (clockwise) from positive angles as shown.

Negative angles



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In the 4th quadrant:

$\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

$\tan(-\theta) = -\tan\theta$

In the 3rd quadrant:

$\sin(-(180° - \theta)) = -\sin\theta$

$\cos(-(180° - \theta)) = -\cos\theta$

$\tan(-(180° - \theta)) = \tan\theta$

In the 2nd quadrant:

$\sin(-(180° + \theta)) = \sin\theta$

$\cos(-(180° + \theta)) = -\cos\theta$

$\tan(-(180° + \theta)) = -\tan\theta$

In the 1st quadrant:

$\sin(-(360° - \theta)) = \sin\theta$

$\cos(-(360° - \theta)) = \cos\theta$

$\tan(-(360° - \theta)) = \tan\theta$

EXAMPLE 3

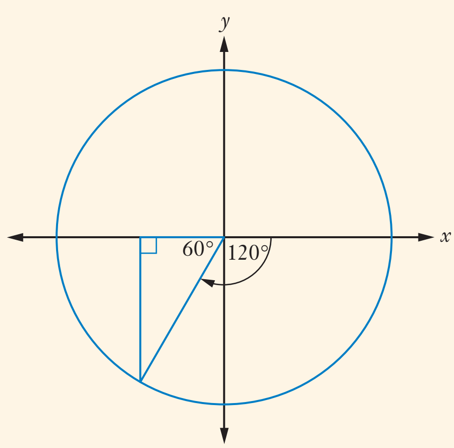
Find the exact value of tan (–120°).

Solution

Moving clockwise around the circle, the angle is in the 3rd quadrant, with $180° - 120° = 60°$ in the triangle.

tan is positive in the 3rd quadrant.

$$\tan(-120°) = \tan 60°$$
$$= \sqrt{3}$$



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Exercise 9.01 Angles of any magnitude

1 Find all quadrants where:

a $\cos \theta > 0$ **b** $\tan \theta > 0$ **c** $\sin \theta > 0$
d $\tan \theta < 0$ **e** $\sin \theta < 0$ **f** $\cos \theta < 0$
g $\sin \theta < 0$ and $\tan \theta > 0$ **h** $\cos \theta < 0$ and $\tan \theta < 0$
i $\cos \theta > 0$ and $\tan \theta < 0$ **j** $\sin \theta < 0$ and $\tan \theta < 0$

2 a Which quadrant is the angle 240° in?
b Find the exact value of cos 240°.

3 a Which quadrant is the angle 315° in?
b Find the exact value of sin 315°.

4 a Which quadrant is the angle 120° in?
b Find the exact value of tan 120°.

5 a Which quadrant is the angle –225° in?
b Find the exact value of sin (–225°).

6 a Which quadrant is the angle –330° in?
b Find the exact value of cos (–330°).

7 Find the exact value of:

a tan 225° **b** cos 315° **c** tan 300° **d** sin 150°
e cos 120° **f** sin 210° **g** cos 330° **h** tan 150°
i sin 300° **j** cos 135°

8 Find the exact value of:

a cos (–225°) **b** cos (–210°) **c** tan (–300°) **d** cos (–150°)
e sin (–60°) **f** tan (–240°) **g** cos (–300°) **h** tan (–30°)
i cos (–45°) **j** sin (–135°)

9 Find the exact value of:

a cos 570° **b** tan 420° **c** sin 480° **d** cos 660°
e sin 690° **f** tan 600° **g** sin 495° **h** cos 405°
i tan 675° **j** sin 390°

10 If $\tan \theta = \frac{3}{4}$ and $\cos \theta < 0$, find $\sin \theta$ and $\cos \theta$ as fractions.

11 Given $\sin \theta = \frac{4}{7}$ and $\tan \theta < 0$, find the exact value of $\cos \theta$ and $\tan \theta$.

12 If $\sin x < 0$ and $\tan x = -\frac{5}{8}$, find the exact value of $\cos x$.

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13 Given $\cos x = \frac{2}{5}$ and $\tan x < 0$, find the exact value of $\sin x$ and $\tan x$.

14 If $\cos x < 0$ and $\sin x > 0$, find $\cos x$ and $\sin x$ in surd form if $\tan x = \frac{5}{7}$.

15 If $\sin \theta = -\frac{4}{9}$ and $270° < \theta < 360°$, find the exact value of $\tan \theta$ and $\cos \theta$.

16 If $\cos x = -\frac{3}{8}$ and $180° < x < 270°$, find the exact value of $\tan x$ and $\sin x$.

17 Given $\sin x = 0.3$ and $\tan x < 0$:

a express $\sin x$ as a fraction

b find the exact value of $\cos x$ and $\tan x$.

18 If $\tan \alpha = -1.2$ and $270° < \alpha < 360°$, find the exact values of $\cos \alpha$ and $\sin \alpha$.

19 Given that $\cos \theta = -0.7$ and $90° < \theta < 180°$, find the exact value of $\sin \theta$ and $\tan \theta$.

20 Simplify:

a $\sin (180° - \theta)$ **b** $\cos (360° - x)$ **c** $\tan (180° + \beta)$

d $\sin (180° + \alpha)$ **e** $\tan (360° - \theta)$ **f** $\sin (-\theta)$

g $\cos (-\alpha)$ **h** $\tan (-x)$

9.02 Trigonometric identities

WS Trigonometric identities

Simplifying trigonometric functions

The reciprocal trigonometric ratios

The **reciprocal trigonometric ratios** are the reciprocals of the sine, cosine and tangent ratios.

The reciprocal trigonometric ratios

Cosecant $\operatorname{cosec} \theta = \frac{1}{\sin \theta} = \frac{\text{hypotenuse}}{\text{opposite}}$

cosec θ can also be written as *csc* θ.

Secant $\sec \theta = \frac{1}{\cos \theta} = \frac{\text{hypotenuse}}{\text{adjacent}}$

Cotangent $\cot \theta = \frac{1}{\tan \theta} = \frac{\text{adjacent}}{\text{opposite}}$

The reciprocal ratios have the same signs as their related ratios in the different quadrants. For example, in the 3rd and 4th quadrants, $\sin \theta < 0$, so $\operatorname{cosec} \theta < 0$.

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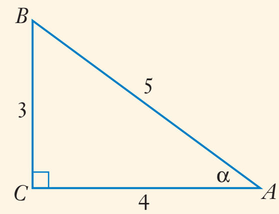
EXAMPLE 4

a Find cosec α, sec α and cot α for this triangle.



b If $\sin\theta = -\frac{2}{7}$ and $\tan > 0$, find the exact ratios of cot θ, sec θ and cosec θ.

c State the quadrants where cosec θ is negative.

Solution

a

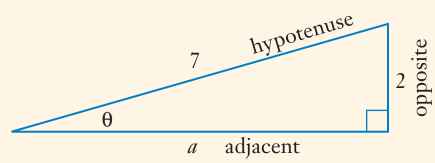
$$\operatorname{cosec}\alpha = \frac{1}{\sin\alpha} = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{5}{3}$$

$$\sec\alpha = \frac{1}{\cos\alpha} = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{5}{4}$$

$$\cot\alpha = \frac{1}{\tan\alpha} = \frac{\text{adjacent}}{\text{opposite}} = \frac{4}{3}$$

b $\sin\theta < 0$ and $\tan\theta > 0$ in the 3rd quadrant. So $\cos\theta < 0$.

By Pythagoras' theorem:



$$7^2 = a^2 + 2^2$$
$$a^2 + 4 = 49$$
$$a^2 = 45$$
$$a = \sqrt{45}$$
$$= 3\sqrt{5}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{\text{adjacent}}{\text{opposite}} = \frac{3\sqrt{5}}{2}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{\text{hypotenuse}}{\text{adjacent}} = -\frac{7}{3\sqrt{5}} = -\frac{7\sqrt{5}}{15}$$

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{\text{hypotenuse}}{\text{opposite}} = -\frac{7}{2}$$

c $\sin\theta < 0$ in the 3rd and 4th quadrants.

So $\operatorname{cosec}\theta < 0$ in the 3rd and 4th quadrants.

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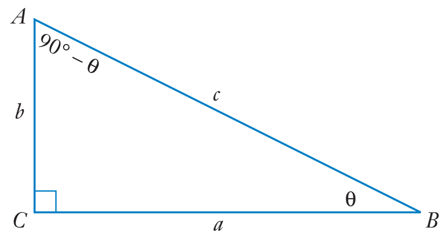
Complementary angles

In $\triangle ABC$ if $\angle B = \theta$ then $\angle A = 90° - \theta$ (by the angle sum of a triangle). $\angle B$ and $\angle A$ are **complementary angles** because they add up to 90°.



$$\sin\theta = \frac{b}{c} \qquad \sin(90° - \theta) = \frac{a}{c}$$

$$\cos\theta = \frac{a}{c} \qquad \cos(90° - \theta) = \frac{b}{c}$$

$$\tan\theta = \frac{b}{a} \qquad \tan(90° - \theta) = \frac{a}{b}$$

$$\sec\theta = \frac{c}{a} \qquad \sec(90° - \theta) = \frac{c}{b}$$

$$\operatorname{cosec}\theta = \frac{c}{b} \qquad \operatorname{cosec}(90° - \theta) = \frac{c}{a}$$

$$\cot\theta = \frac{a}{b} \qquad \cot(90° - \theta) = \frac{b}{a}$$

Notice the pairs of trigonometric ratios that are equal:

Complementary angle results

$\sin\theta = \cos(90° - \theta)$ $\quad$ $\tan\theta = \cot(90° - \theta)$ $\quad$ $\sec\theta = \operatorname{cosec}(90° - \theta)$

$\cos\theta = \sin(90° - \theta)$ $\quad$ $\cot\theta = \tan(90° - \theta)$ $\quad$ $\operatorname{cosec}\theta = \sec(90° - \theta)$

EXAMPLE 5

a Simplify $\tan 50° - \cot 40°$.

b Find the value of m if $\sec 55° = \operatorname{cosec}(2m - 15)°$.

Solution

a
$$\begin{aligned}\tan 50° - \cot 40° &= \tan 50° - \cot(90° - 50°)\\ &= \tan 50° - \tan 50°\\ &= 0\end{aligned}$$

b
$$\begin{aligned}\sec 55° &= \operatorname{cosec}(90° - 55°)\\ &= \operatorname{cosec} 35°\end{aligned}$$

$$\begin{aligned}\text{So } 2m - 15 &= 35\\ 2m &= 50\\ m &= 25\end{aligned}$$

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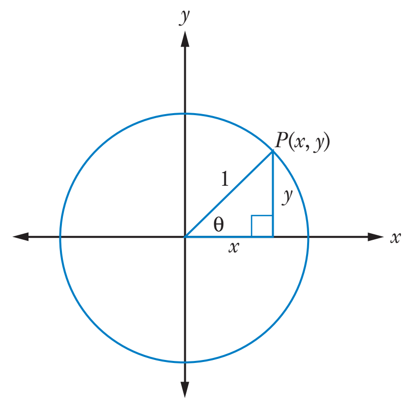
The tangent identity



In the work on angles of any magnitude, we saw that $\sin\theta = y$, $\cos\theta = x$ and $\tan\theta = \dfrac{y}{x}$.
From this we get the following trigonometric identities:

The tangent identity

For any value of θ:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

An **identity** is an equation that shows the equivalence of 2 algebraic expressions for all values of the variables, for example, $a^2 - b^2 = (a + b)(a - b)$ is an identity.

EXAMPLE 6

Simplify $\sin\theta\cot\theta$.

Solution

$$\begin{aligned}\sin\theta\cot\theta &= \sin\theta \times \frac{\cos\theta}{\sin\theta}\\ &= \cos\theta\end{aligned}$$

The Pythagorean identities

The unit circle above has equation $x^2 + y^2 = 1$, because of Pythagoras' theorem.

But $\sin\theta = y$ and $\cos\theta = x$, so

$(\cos\theta)^2 + (\sin\theta)^2 = 1$

A shorter way of writing this is:

$\cos^2\theta + \sin^2\theta = 1$

This formula is called a Pythagorean identity because it is based on Pythagoras' theorem in the unit circle.

There are 2 other identities that can be derived from this identity.

Dividing each term by $\cos^2\theta$:

$$\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$1 + \tan^2\theta = \sec^2\theta$$

Dividing each term by $\sin^2\theta$:

$$\frac{\cos^2\theta}{\sin^2\theta} + \frac{\sin^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$\cot^2\theta + 1 = \operatorname{cosec}^2\theta$$

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Pythagorean identities

For any value of θ:

$$\cos^2\theta + \sin^2\theta = 1$$
$$1 + \tan^2\theta = \sec^2\theta$$
$$1 + \cot^2\theta = \text{cosec}^2\theta$$

$\cos^2\theta + \sin^2\theta = 1$ can also be rearranged to give:

$$\cos^2\theta = 1 - \sin^2\theta \text{ or}$$
$$\sin^2\theta = 1 - \cos^2\theta$$

EXAMPLE 7

Prove that:

a $\cot x + \tan x = \text{cosec } x \sec x$

b $\dfrac{1-\cos x}{\sin^2 x} = \dfrac{1}{1+\cos x}$

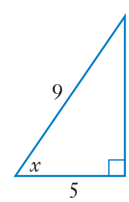
Solution

a
$$\begin{aligned}
\text{LHS} &= \cot x + \tan x \\
&= \frac{\cos x}{\sin x} + \frac{\sin x}{\cos x} \\
&= \frac{\cos^2 x + \sin^2 x}{\sin x \cos x} \\
&= \frac{1}{\sin x \cos x} \\
&= \frac{1}{\sin x} \times \frac{1}{\cos x} \\
&= \text{cosec } x \sec x \\
&= \text{RHS}
\end{aligned}$$
$\therefore \cot x + \tan x = \text{cosec } x \sec x$

b
$$\begin{aligned}
\text{LHS} &= \frac{1-\cos x}{\sin^2 x} \\
&= \frac{1-\cos x}{1-\cos^2 x} \\
&= \frac{1-\cos x}{(1+\cos x)(1-\cos x)} \\
&= \frac{1}{1+\cos x} \\
&= \text{RHS}
\end{aligned}$$
$$\therefore \frac{1-\cos x}{\sin^2 x} = \frac{1}{1+\cos x}$$

Exercise 9.02 Trigonometric identities

1 For this triangle, find the exact ratios of sec x, cot x and cosec x.



2 If $\sin\theta = \dfrac{5}{13}$, find cosec θ, sec θ and cot θ.

3 If $\cos\theta = \dfrac{4}{7}$, find exact values of cosec θ, sec θ and cot θ.

4 If $\sec\theta = -\dfrac{6}{5}$ and $\sin\theta > 0$, find exact values of tan θ, cosec θ and cot θ.

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5 If $\cot \theta = 0.6$ and $\operatorname{cosec} \theta < 0$, find the exact values of $\sin \theta$, $\operatorname{cosec} \theta$, $\tan \theta$ and $\sec \theta$.

6 Show $\sin 67° = \cos 23°$.

7 Show $\sec 82° = \operatorname{cosec} 8°$.

8 Show $\tan 48° = \cot 42°$.

9 Simplify:

a $\cos 61° + \sin 29°$

b $\sec \theta - \operatorname{cosec} (90° - \theta)$

c $\tan 70° + \cot 20° - 2 \tan 70°$

d $\dfrac{\sin 55°}{\cos 35°}$

e $\dfrac{\cot 25° + \tan 65°}{\cot 25°}$

10 Find the value of x if $\sin 80° = \cos (90 - x)°$

11 Find the value of y if $\tan 22° = \cot (90 - y)°$

12 Find the value of p if $\cos 49° = \sin (p + 10)°$

13 Find the value of b if $\sin 35° = \cos (b + 30)°$

14 Find the value of t if $\cot (2t + 5)° = \tan (3t - 15)°$

15 Find the value of k if $\tan (15 - k)° = \cot (2k + 60)°$

16 Simplify:

a $\tan \theta \cos \theta$

b $\tan \theta \operatorname{cosec} \theta$

c $\sec x \cot x$

d $1 - \sin^2 x$

e $\sqrt{1-\cos^2 \circ}$

f $\cot^2 x + 1$

g $1 + \tan^2 x$

h $\sec^2 \theta - 1$

i $5 \cot^2 \theta + 5$

j $\dfrac{1}{\operatorname{cosec}^2 x}$

k $\sin^2 \alpha \operatorname{cosec}^2 \alpha$

l $\cot \theta - \cot \theta \cos^2 \theta$

17 Prove that:

a $\cos^2 x - 1 = -\sin^2 x$

b $\sec \theta + \tan \theta = \dfrac{1+\sin \circ}{\cos \circ}$

c $3 + 3 \tan^2 \alpha = \dfrac{3}{1-\sin^2 \circ}$

d $\sec^2 x - \tan^2 x = \operatorname{cosec}^2 x - \cot^2 x$

e $(\sin x - \cos x)^3 = \sin x - \cos x - 2 \sin^2 x \cos x + 2 \sin x \cos^2 x$

f $\cot \theta + 2 \sec \theta = \dfrac{1-\sin^2 \circ + 2\sin \circ}{\sin \circ \cos \circ}$

g $\cos^2 (90° - \theta) \cot \theta = \sin \theta \cos \theta$

h $(\operatorname{cosec} x + \cot x)(\operatorname{cosec} x - \cot x) = 1$

i $\dfrac{1-\sin^2 \circ \cos^2 \circ}{\cos^2 \circ} = \tan^2 \theta + \cos^2 \theta$

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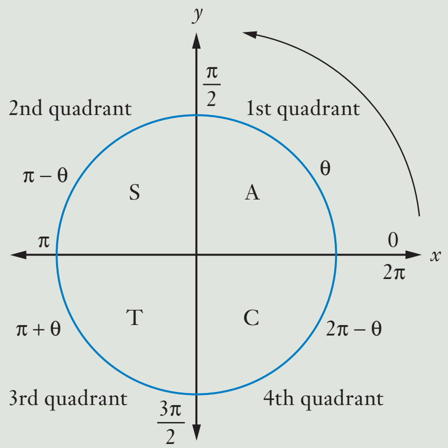
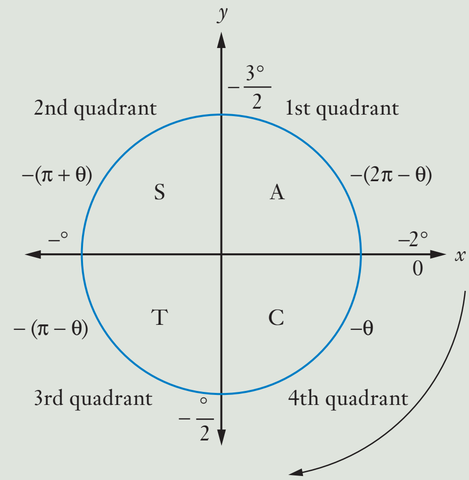
9.03 Radians

The rules and formulas learned in this chapter can also be expressed in radians, which we learned about in Chapter 4, *Trigonometry*.

WS
Radians of any magnitude

ASTC rule





In the 2nd quadrant:

$\sin(\pi - \theta) = \sin\theta$

$\cos(\pi - \theta) = -\cos\theta$

$\tan(\pi - \theta) = -\tan\theta$

In the 3rd quadrant:

$\sin(\pi + \theta) = -\sin\theta$

$\cos(\pi + \theta) = -\cos\theta$

$\tan(\pi + \theta) = \tan\theta$

In the 4th quadrant:

$\sin(2\pi - \theta) = -\sin\theta$

$\cos(2\pi - \theta) = \cos\theta$

$\tan(2\pi - \theta) = -\tan\theta$

In the 4th quadrant:

$\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

$\tan(-\theta) = -\tan\theta$

In the 3rd quadrant:

$\sin(-(\pi - \theta)) = -\sin\theta$

$\cos(-(\pi - \theta)) = -\cos\theta$

$\tan(-(\pi - \theta)) = \tan\theta$

In the 2nd quadrant:

$\sin(-(\pi + \theta)) = \sin\theta$

$\cos(-(\pi + \theta)) = -\cos\theta$

$\tan(-(\pi + \theta)) = -\tan\theta$

In the 1st quadrant:

$\sin(-(2\pi - \theta)) = \sin\theta$

$\cos(-(2\pi - \theta)) = \cos\theta$

$\tan(-(2\pi - \theta)) = \tan\theta$

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EXAMPLE 8

Find the exact value of:

a $\sin \frac{5\pi}{4}$ **b** $\cos \frac{11\pi}{6}$

Solution

a $\frac{5\pi}{4} = \frac{4\pi}{4} + \frac{\pi}{4}$

$= \pi + \frac{\pi}{4}$

in the 3rd quadrant, so $\sin \theta < 0$.

$$\sin\left(\frac{5\pi}{4}\right) = \sin\left(\pi + \frac{\pi}{4}\right)$$
$$= -\sin\frac{\pi}{4}$$
$$= -\frac{1}{\sqrt{2}}$$

b $\frac{11\pi}{6} = \frac{12\pi}{6} - \frac{\pi}{6}$

$= 2\pi - \frac{\pi}{6}$

in the 4th quadrant, so $\cos \theta > 0$.

$$\cos\left(\frac{11\pi}{6}\right) = \cos\left(2\pi - \frac{\pi}{6}\right)$$
$$= \cos\frac{\pi}{6}$$
$$= \frac{\sqrt{3}}{2}$$

Exercise 9.03 Radians

1 Find the exact value of each expression.

a $\operatorname{cosec} \frac{\pi}{4}$ **b** $\sec \frac{\pi}{6}$ **c** $\cot \frac{\pi}{3}$

d $\dfrac{\sin \frac{\pi}{3}}{\cos \frac{\pi}{3}}$ **e** $1 - \cos^2 \frac{\pi}{4}$ **f** $\tan \frac{\pi}{3} \cos \frac{\pi}{3}$

g $\sqrt{1 + \tan^2 \frac{\pi}{4}}$ **h** $\operatorname{cosec}^2 \frac{\pi}{6} - 1$ **i** $\dfrac{\cot \frac{\pi}{5} + \tan \frac{3\pi}{10}}{\cot \frac{\pi}{5}}$

2 **a** Show that $\frac{3\pi}{4} = \pi - \frac{\pi}{4}$.

b In which quadrant is the angle $\frac{3\pi}{4}$?

c Find the exact value of $\cos \frac{3\pi}{4}$.

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3 **a** Show that $\frac{5\pi}{6} = \pi - \frac{\pi}{6}$.

b In which quadrant is the angle $\frac{5\pi}{6}$?

c Find the exact value of $\sin \frac{5\pi}{6}$.

4 **a** Show that $\frac{7\pi}{4} = 2\pi - \frac{\pi}{4}$.

b In which quadrant is the angle $\frac{7\pi}{4}$?

c Find the exact value of $\tan \frac{7\pi}{4}$.

5 **a** Show that $\frac{4\pi}{3} = \pi + \frac{\pi}{3}$.

b In which quadrant is the angle $\frac{4\pi}{3}$?

c Find the exact value of $\cos \frac{4\pi}{3}$.

6 **a** Show that $\frac{5\pi}{3} = 2\pi - \frac{\pi}{3}$.

b In which quadrant is the angle $\frac{5\pi}{3}$?

c Find the exact value of $\sin \frac{5\pi}{3}$.

7 **a** **i** Show that $\frac{13\pi}{6} = 2\pi + \frac{\pi}{6}$.

ii In which quadrant is the angle $\frac{13\pi}{6}$?

iii Find the exact value of $\cos \frac{13\pi}{6}$.

b Find the exact value of:

i $\sin \frac{9\pi}{4}$ **ii** $\tan \frac{7\pi}{3}$ **iii** $\cos \frac{11\pi}{4}$

iv $\tan \frac{19\pi}{6}$ **v** $\sin \frac{10\pi}{3}$

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8 Copy and complete each table with exact values.

a

	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	$\frac{7\pi}{3}$	$\frac{8\pi}{3}$	$\frac{10\pi}{3}$	$\frac{11\pi}{3}$
sin								
cos								
tan								

b

	$\frac{\pi}{4}$	$\frac{3\pi}{4}$	$\frac{5\pi}{4}$	$\frac{7\pi}{4}$	$\frac{9\pi}{4}$	$\frac{11\pi}{4}$	$\frac{13\pi}{4}$	$\frac{15\pi}{4}$
sin								
cos								
tan								

c

	$\frac{\pi}{6}$	$\frac{5\pi}{6}$	$\frac{7\pi}{6}$	$\frac{11\pi}{6}$	$\frac{13\pi}{6}$	$\frac{17\pi}{6}$	$\frac{19\pi}{6}$	$\frac{23\pi}{6}$
sin								
cos								
tan								

9 Copy and complete the table where possible.

	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π	$\frac{7\pi}{2}$	4π
sin									
cos									
tan									

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9.04 Trigonometric functions

INVESTIGATION

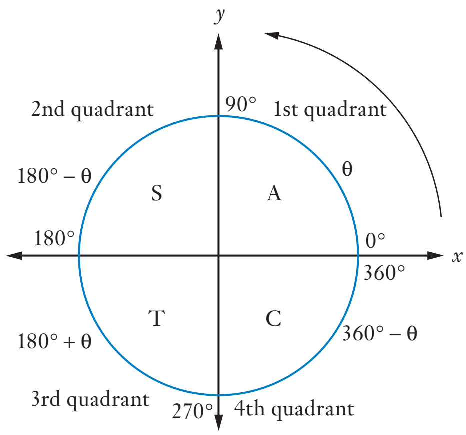
TRIGONOMETRIC RATIOS OF 0°, 90°, 180°, 270° AND 360°

Remember the results from the unit circle:

$\sin \theta = y$

$\cos \theta = x$

$\tan \theta = \dfrac{y}{x}$



1 Angle 0° is at the point (1, 0) on the unit circle. Use the circle results to find sin 0°, cos 0° and tan 0°.

2 Angle 90° is at the point (0, 1). Use the circle results to find sin 90°, cos 90° and tan 90°. Discuss the result for tan 90° and why this happens.

3 Angle 180° is at the point (–1, 0). Find sin 180°, cos 180° and tan 180°.

4 Angle 270° is at the point (0, –1). Find sin 270°, cos 270° and tan 270°. Discuss the result for tan 270° and why this happens.

5 What are the results for sin 360°, cos 360° and tan 360°? Why?

6 Check these results on your calculator.

WS Sine and cosine curves

WS Trigonometric graphs

Trigonometric graphs match-up

WS Sketching periodic functions: amplitude and period

WS Sketching periodic functions: phase and vertical shift

Amplitude and period

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The sine function

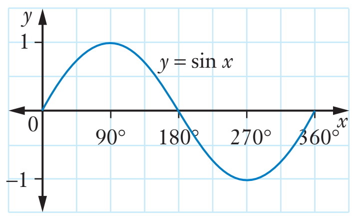
Using all the results from the investigation, we can draw up a table of values for $y = \sin x$.

x	0°	90°	180°	270°	360°
y	0	1	0	–1	0

We could add in all the exact value results we know for a more accurate graph. Remember that $\sin x$ is positive in the 1st and 2nd quadrants and negative in the 3rd and 4th quadrants.

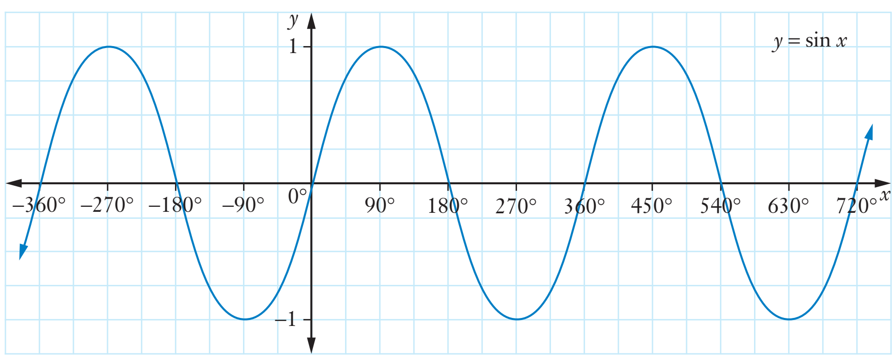
x	0°	30°	45°	60°	90°	120°	135°	150°	180°	210°	225°	240°	270°	300°	315°	330°	360°
y	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	–1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$	0

Drawing the graph gives a smooth 'wave' curve.



As we go around the unit circle and graph the y values of the points on the circle, the graph should repeat itself every 360°.

$y = \sin x$ has domain $(-\infty, \infty)$ and range $[-1, 1]$. It is an odd function.



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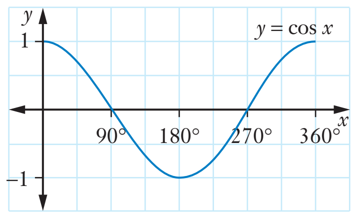
The cosine function

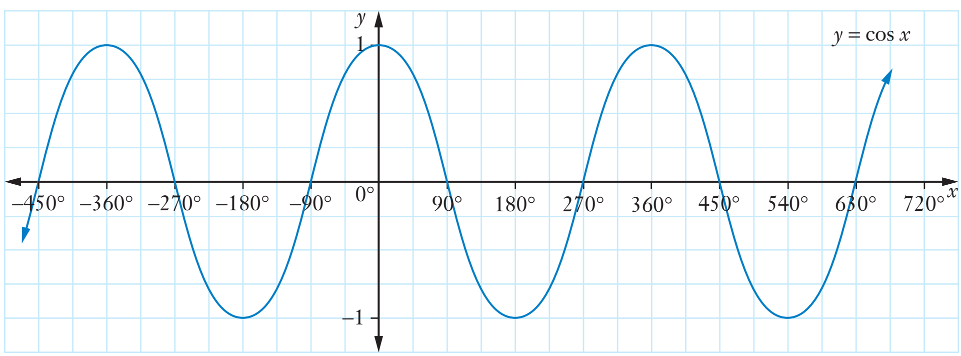
Similarly for $y = \cos x$, which is positive in the 1st and 4th quadrants and negative in the 2nd and 3rd quadrants. Its graph has the same shape as the graph of the sine function.

x	0°	30°	45°	60°	90°	120°	135°	150°	180°	210°	225°	240°	270°	300°	315°	330°	360°
y	0	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	1	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	0	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$	-1	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	0



As we go around the unit circle and graph the x values of the points on the circle, the graph should repeat itself every 360°.

$y = \cos x$ has domain $(-\infty, \infty)$ and range $[-1, 1]$. It is an even function.



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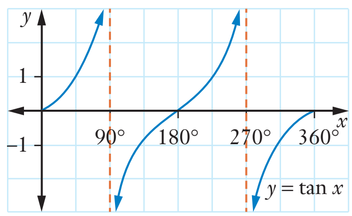
The tangent function

$y = \tan x$ is positive in the 1st and 3rd quadrants and negative in the 2nd and 4th quadrants.

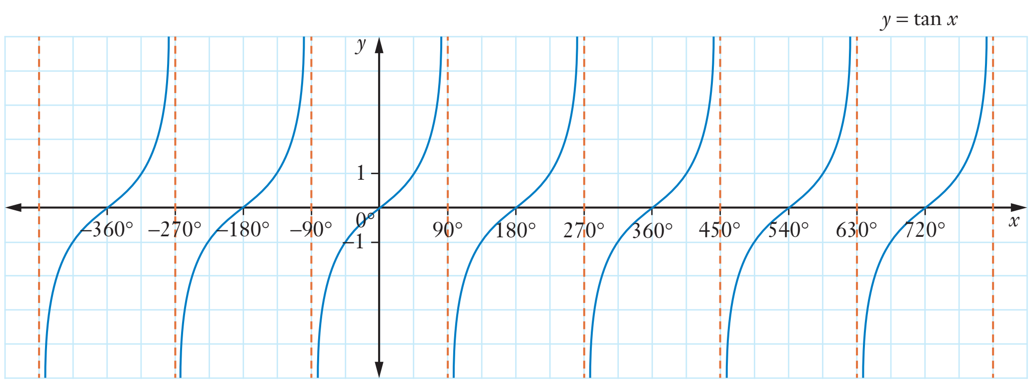
It is also undefined for 90° and 270° so there are vertical asymptotes at those x values, where the function is discontinuous.

x	0°	30°	45°	60°	90°	120°	135°	150°	180°	210°	225°	240°	270°	300°	315°	330°	360°
y	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	–	$-\sqrt{3}$	–1	$-\frac{1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	–	$-\sqrt{3}$	–1	$-\frac{1}{\sqrt{3}}$	0



As we go around the unit circle and graph the values of $\frac{y}{x}$ of the points on the circle, the graph repeats itself every 180°.

$y = \tan x$ has domain $(-\infty, \infty)$ except for 90°, 270°, 540°, … (odd multiples of 90°) and range $(-\infty, \infty)$. It is an odd function.



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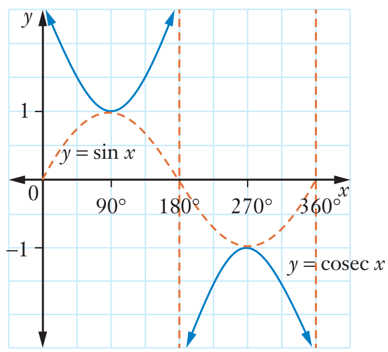
The cosecant function

$$\operatorname{cosec} x = \frac{1}{\sin x}$$

Each y value of $y = \operatorname{cosec} x$ will be the reciprocal of $y = \sin x$. Because $\sin x = 0$ at $x = 0°, 180°, 360°, \ldots$, $y = \operatorname{cosec} x$ will have vertical asymptotes at those values.

We can use a table of values and explore the limits as x approaches any asymptotes.

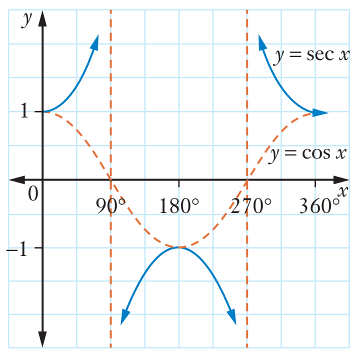
x	0°	30°	45°	60°	90°	120°	135°	150°	180°	210°	225°	240°	270°	300°	315°	330°	360°
y	–	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	–	−2	$-\sqrt{2}$	$-\frac{2}{\sqrt{3}}$	−1	$-\frac{2}{\sqrt{3}}$	$-\sqrt{2}$	−2	–



The secant function

$\sec x = \dfrac{1}{\cos x}$, so each y value of $y = \sec x$ will be the reciprocal of $y = \cos x$. Because $\cos x = 0$ at $x = 90°, 270°, 450°, \ldots$, $y = \sec x$ will have vertical asymptotes at those values.

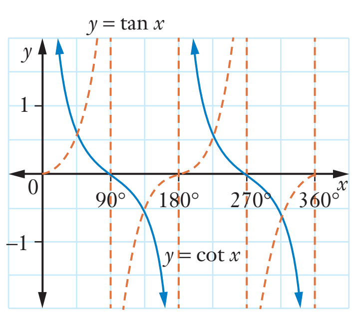
x	0°	30°	45°	60°	90°	120°	135°	150°	180°	210°	225°	240°	270°	300°	315°	330°	360°
y	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	–	−2	$-\sqrt{2}$	$-\frac{2}{\sqrt{3}}$	−1	$-\frac{2}{\sqrt{3}}$	$-\sqrt{2}$	−2	–	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1



The cotangent function

$\cot x = \dfrac{1}{\tan x}$, so each y value of $y = \cot x$ will be the reciprocal of $y = \tan x$. Because $\tan x = 0$ at $x = 0°, 180°, 360°, \ldots$, $y = \cot x$ will have vertical asymptotes at those values. Also, because $\tan x$ has asymptotes at $x = 90°, 270°, 450°, \ldots$, $y = \cot x = 0$ and there are x-intercepts at those values.

x	0°	30°	45°	60°	90°	120°	135°	150°	180°	210°	225°	240°	270°	300°	315°	330°	360°
y	–	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$-\frac{1}{\sqrt{3}}$	−1	$-\sqrt{3}$	–	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$-\frac{1}{\sqrt{3}}$	−1	$-\sqrt{3}$	–



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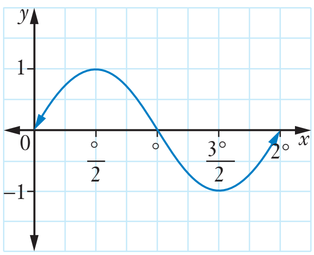
It is more practical to express the trigonometric functions in terms of radians (not degrees), so here are the graphs in radians:

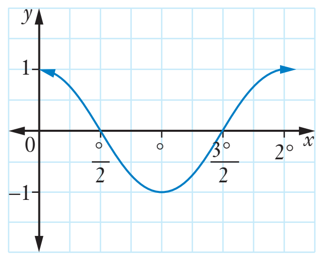
$y = \sin x$



Domain $(-\infty, \infty)$, range $[-1, 1]$

Odd function

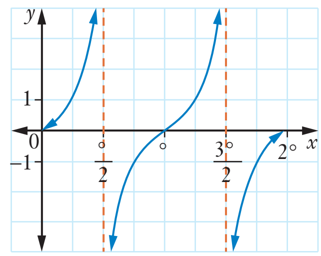
$y = \cos x$



Domain $(-\infty, \infty)$, range $[-1, 1]$

Even function

$y = \tan x$

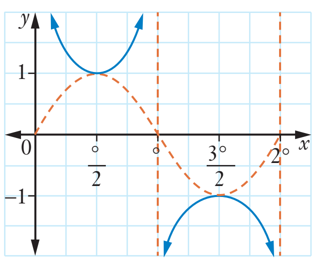


Domain $(-\infty, \infty)$

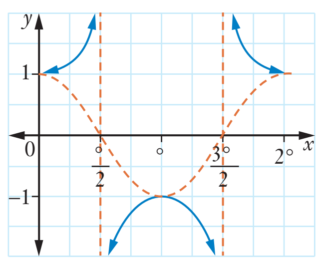
except for $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

(odd multiples of $\frac{\pi}{2}$),
range $(-\infty, \infty)$
Odd function

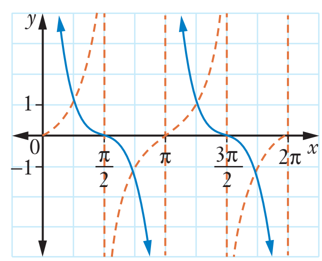
$y = \operatorname{cosec} x$



$y = \sec x$



$y = \cot x$



Properties of the trigonometric functions

All the trigonometric functions have graphs that repeat at regular intervals, so they are called **periodic functions**. The **period** is the length of one cycle of a periodic function on the x-axis, before the function repeats itself.

The **centre** of a periodic function is its mean value and is equidistant from the maximum and minimum values. The mean value of $y = \sin x$, $y = \cos x$ and $y = \tan x$ is 0, represented by the x-axis.

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The **amplitude** is the height from the centre of a periodic function to the maximum or minimum values (peaks and troughs of its graph respectively). The range of $y = \sin x$ and $y = \cos x$ is $[-1, 1]$.

$y = \sin x$ has period 2π and amplitude 1.

$y = \cos x$ has period 2π and amplitude 1.

$y = \tan x$ has period π and no amplitude.

INVESTIGATION

TRANSFORMING TRIGONOMETRIC GRAPHS

Use a graphics calculator or graphing software to draw the graphs of trigonometric functions with different values.

1 Graphs in the form $y = k \sin x$, $y = k \cos x$ and $y = k \tan x$ where $k = \ldots, -3, -2, -1, 2, 3, \ldots$
2 Graphs in the form $y = \sin ax$, $y = \cos ax$ and $y = \tan ax$ where $a = \ldots, -3, -2, -1, 2, 3, \ldots$
3 Graphs in the form $y = \sin x + c$, $y = \cos x + c$ and $y = \tan x + c$ where $c = \ldots, -3, -2, -1, 2, 3, \ldots$
4 Graphs in the form $y = \sin (x + b)$, $y = \cos (x + b)$ and $y = \tan (x + b)$ where $b = \ldots, \pm\frac{\circ}{2}, \pm\pi, \pm\frac{\circ}{4}, \ldots$

Can you see patterns? Could you predict what different graphs look like?

Now we shall examine more general trigonometric functions of the form $y = k \sin ax$, $y = k \cos ax$ and $y = k \tan ax$, where k and a are constants.

Period and amplitude of trigonometric functions

$y = k \sin ax$ has amplitude k and period $\frac{2\pi}{a}$.

$y = k \cos ax$ has amplitude k and period $\frac{2\pi}{a}$.

$y = k \tan ax$ has no amplitude and has period $\frac{\pi}{a}$.

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EXAMPLE 9

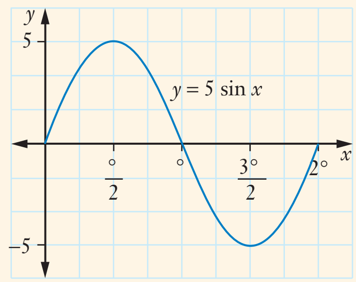
a Sketch each function in the domain $[0, 2\pi]$.

i $y = 5 \sin x$ **ii** $y = \sin 4x$ **iii** $y = 5 \sin 4x$

b Sketch the graph of $y = 2 \tan \frac{x}{2}$ for $[0, 2\pi]$.

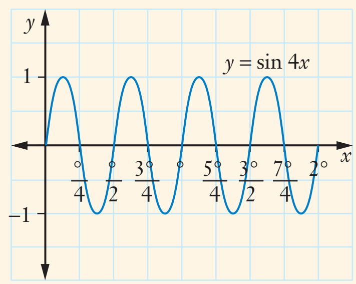
Solution

a **i** The graph of $y = 5 \sin x$ has y values that are 5 times as much as $y = \sin x$, so this function has amplitude 5 and period 2π. We draw one period of the sine 'shape' between ± 5.

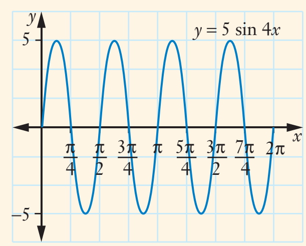


ii The graph $y = \sin 4x$ has amplitude 1 and period $\frac{2\pi}{4} = \frac{\pi}{2}$.

The curve repeats every $\frac{\pi}{2}$, so in the domain $[0, 2\pi]$ there will be 4 repetitions. The '4' in $\sin 4x$ compresses the graph of $y = \sin x$ horizontally.



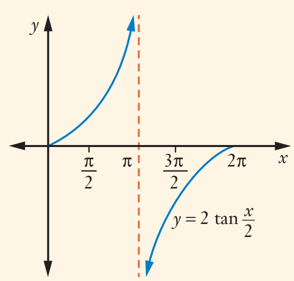
iii The graph $y = 5 \sin 4x$ has amplitude 5 and period $\frac{\pi}{2}$. It is a combination of graphs **i** and **ii**.



b $y = 2 \tan \frac{x}{2}$ has no amplitude.

$$\text{Period} = \frac{\pi}{\frac{1}{2}} = 2\pi$$

So there will be one period in the domain $[0, 2\pi]$.



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The graphs of trigonometric functions can change their **phase**, a shift to the left or right.

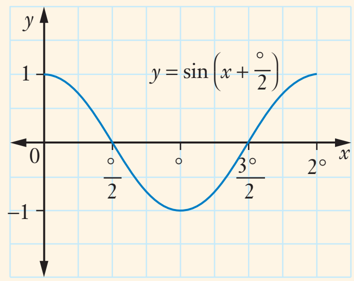
Phase shift of trigonometric functions

$y = \sin(x + b)$, $y = \cos(x + b)$ and $y = \tan(x + b)$ have phase b, which is a shift b units from $y = \sin x$, $y = \cos x$ and $y = \tan x$ respectively, to the left if $b > 0$ and to the right if $b < 0$.

EXAMPLE 10

Sketch the graph of $f(x) = \sin\left(x + \frac{\circ}{2}\right)$ for $[0, 2\pi]$.

Solution



Amplitude = 1

$$\text{Period} = \frac{2\pi}{1} = 2\pi$$

$$\text{Phase: } b = \frac{\pi}{2}$$

This is the graph of $y = \sin x$ moved $\frac{\pi}{2}$ units to the left. If you're unsure how the phase affects the graph, draw a table of values.

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
y	1	0	–1	0	1

The graphs of trigonometric functions can change their centre, a shift up or down.

Centre of trigonometric functions

$y = \sin x + c$, $y = \cos x + c$ and $y = \tan x + c$ have centre c, which is a shift up from $y = \sin x$, $y = \cos x$ and $y = \tan x$ respectively if $c > 0$ and a shift down if $c < 0$.

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EXAMPLE 11

Graphing the cosine function

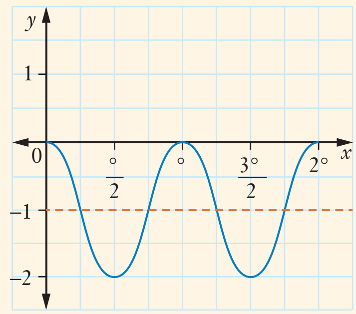
Sketch the graph of $y = \cos 2x - 1$ in the domain $[0, 2\pi]$.

Solution

Amplitude = 1, period $\dfrac{2\pi}{2} = \pi$.

$c = -1$ so the centre of the graph moves down 1 unit to -1.

Instead of moving between -1 and 1, the graph moves between -2 and 0.



General trigonometric functions

	Amplitude	Period	Phase	Centre
$y = k \sin [a(x + b)] + c$	k	$\dfrac{2\pi}{a}$	b Shift left if $b > 0$ Shift right if $b < 0$	$y = c$ Shift up if $c > 0$ Shift down if $c < 0$
$y = k \cos [a(x + b)] + c$	k	$\dfrac{2\pi}{a}$		
$y = k \tan [a(x + b)] + c$	No amplitude	$\dfrac{\pi}{a}$		

EXAMPLE 12

For the function $y = 3 \cos (2x - \pi)$, find:

a the amplitude **b** the period **c** the phase

Solution

$y = 3 \cos (2x - \pi)$

$= 3 \cos \left[2\left(x - \dfrac{\pi}{2}\right)\right]$

a Amplitude = 3

b Period $= \dfrac{2\pi}{2}$
$= \pi$

c Phase $= \dfrac{\pi}{2}$ units

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EXAMPLE 13

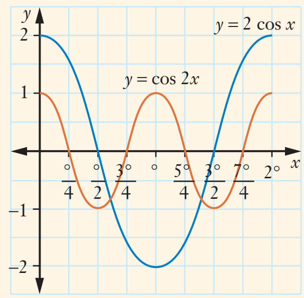
a Sketch the graph of $y = 2\cos x$ and $y = \cos 2x$ on the same set of axes for $[0, 2\pi]$.

b Hence, sketch the graph of $y = \cos 2x + 2\cos x$ for $[0, 2\pi]$.

Solution

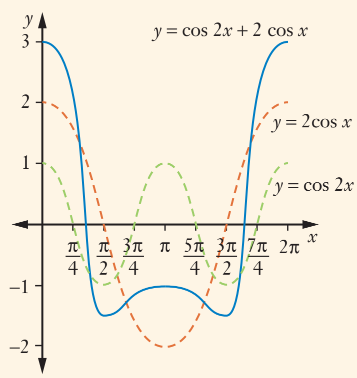
a $y = 2\cos x$ has amplitude 2 and period 2π.

$y = \cos 2x$ has amplitude 1 and period $\frac{2\pi}{2}$ or π.



b Add y values on the graph, using a table of values if more accuracy is needed.

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
$\cos 2x$	1	0	–1	0	1	0	–1	0	1
$2\cos x$	2	$\sqrt{2}$	0	$-\sqrt{2}$	–2	$-\sqrt{2}$	0	$\sqrt{2}$	2
$\cos 2x + 2\cos x$	3	$\sqrt{2}$	–1	$-\sqrt{2}$	–1	$-\sqrt{2}$	–1	$\sqrt{2}$	3



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Exercise 9.04 Trigonometric functions

1 **a** Sketch the graph of $f(x) = \cos x$ in the domain $[0, 2\pi]$.

b Sketch the graph of $y = -f(x)$ in the same domain.

2 Sketch the graph of each function in the domain $[0, 2\pi]$.

a $f(x) = 2 \sin x$ **b** $y = 1 + \sin x$ **c** $y = 2 - \sin x$

d $f(x) = -3 \cos x$ **e** $y = 4 \sin x$ **f** $f(x) = \cos x + 3$

g $y = 5 \tan x$ **h** $f(x) = \tan x + 3$ **i** $y = 1 - 2 \tan x$

3 Sketch the graph of each function in the domain $[0, 2\pi]$.

a $y = \cos 2x$ **b** $y = \tan 2x$ **c** $y = \sin 3x$

d $f(x) = 3 \cos 4x$ **e** $y = 6 \cos 3x$ **f** $y = \tan \frac{x}{2}$

g $f(x) = 2 \tan 3x$ **h** $y = 3 \cos \frac{x}{2}$ **i** $y = 2 \sin \frac{x}{2}$

4 Sketch the graph of each function in the domain $[-\pi, \pi]$:

a $y = -\sin 2x$ **b** $y = 7 \cos 4x$ **c** $f(x) = -\tan 4x$

d $y = 5 \sin 4x$ **e** $f(x) = 2 \cos 2x$ **f** $f(x) = 3 \tan x - 1$

5 Sketch the graph of $y = 8 \sin \frac{x}{2}$ in the domain $[0, 4\pi]$.

6 Sketch over the interval $[0, 2\pi]$ the graph of:

a $y = \sin (x + \pi)$ **b** $y = \tan\left(x + \frac{\circ}{2}\right)$ **c** $f(x) = \cos (x - \pi)$

d $y = 3 \sin\left(x - \frac{\circ}{2}\right)$ **e** $f(x) = 2 \cos\left(x + \frac{\circ}{2}\right)$ **f** $y = 4 \sin\left(2x + \frac{\circ}{2}\right)$

g $y = \cos\left(x - \frac{\circ}{4}\right)$ **h** $y = \tan\left(x + \frac{\circ}{4}\right)$

7 Sketch over the interval $[-2, 2]$ the graph of:

a $y = \sin \pi x$ **b** $y = 3 \cos 2\pi x$

8 For each function, find:

i the amplitude **ii** the period **iii** the centre **iv** the phase

a $y = 5 \sin 2x$ **b** $f(x) = -\cos (x - \pi)$ **c** $y = 2 \tan (4x) - 2$

d $y = 3 \sin\left(x + \frac{\pi}{4}\right) + 1$ **e** $y = 8 \cos (\pi x - 2) - 3$ **f** $f(x) = 3 \tan\left(5x + \frac{\pi}{2}\right) + 2$

9 Find the domain and range of each function.

a $y = 4 \sin x - 1$ **b** $f(x) = -3 \cos 5x + 7$

10 Sketch in the domain $[0, 2\pi]$ the graphs of:

a $y = \sin x$ and $y = \sin 2x$ on the same set of axes

b $y = \sin x + \sin 2x$

11 Sketch for the interval $[0, 2\pi]$ the graphs of:

a $y = 2 \cos x$ and $y = 3 \sin x$ on the same set of axes

b $y = 2 \cos x + 3 \sin x$

12 By sketching the graphs of $y = \cos x$ and $y = \cos 2x$ on the same set of axes for $[0, 2\pi]$, sketch the graph of $y = \cos 2x - \cos x$.

13 Sketch the graph of $y = \cos x + \sin x$.

9.05 Trigonometric equations

WS Trigonometric equations

EXAMPLE 14

Solve each equation for $[0°, 360°]$.

a $\sin x = 0.34$ **b** $\cos x = \dfrac{\sqrt{3}}{2}$ **c** $\tan \theta = -1$

Solution

Make sure that your calculator is in **degrees mode**. Check your solution by substituting back into the equation.

a 0.34 is positive and $\sin x > 0$ in 1st and 2nd quadrants.

$\sin x = 0.34$

$x \approx 19°53', 180° - 19°53'$

$= 19°53', 160°7'$

19°53′ is the **principal solution** but there is another solution in the 2nd quadrant.

b $\cos x > 0$ in the 1st and 4th quadrants.

$\cos x = \dfrac{\sqrt{3}}{2}$

$x = 30°, 360° - 30°$

$= 30°, 330°$

c $\tan \theta < 0$ in the 2nd and 4th quadrants.

For $\tan \theta = -1$

$\theta = 180° - 45°, 360° - 45°$

$= 135°, 315°$

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EXAMPLE 15

Solve $\tan x = \sqrt{3}$ for $[-180°, 180°]$.

Solution

In the domain $[-180°, 180°]$, we use positive angles for $0° \le x \le 180°$ and negative angles for $-180° \le x \le 0°$.

$\tan > 0$ in the 1st and 3rd quadrants.

$$\begin{aligned} \tan x &= \sqrt{3} \\ x &= 60°, -(180° - 60°) \\ &= 60°, -120° \end{aligned}$$

EXAMPLE 16

Solve $2\sin^2 x - 1 = 0$ for $0° \le x \le 360°$.

Solution

$$\begin{aligned} 2\sin^2 x - 1 &= 0 \\ 2\sin^2 x &= 1 \\ \sin^2 x &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \sin x &= \pm\frac{\sqrt{1}}{\sqrt{2}} \\ &= \pm\frac{1}{\sqrt{2}} \end{aligned}$$

Since the ratio could be positive or negative, there are solutions in all 4 quadrants.

$$\begin{aligned} x &= 45°, 180° - 45°, 180° + 45°, 360° - 45° \\ &= 45°, 135°, 225°, 315° \end{aligned}$$

If we are solving an equation involving $2x$ or $3x$, for example, we need to change the domain to find all possible solutions.

EXAMPLE 17

Solve $2\sin 2x - 1 = 0$ for $[0°, 360°]$.

Solution

Notice that the angle is $2x$ but the domain is for x.

If $\quad 0° \le x \le 360°$

then $0° \le 2x \le 720°$

This means that we can find the solutions by going around the circle twice.

$$2\sin 2x - 1 = 0$$
$$2\sin 2x = 1$$
$$\sin 2x = \frac{1}{2}$$

Sin is positive in the 1st and 2nd quadrants.

First time around the circle: 1st quadrant is θ and the 2nd quadrant is 180° – θ.

Second time around the circle: add 360° to θ and 180° – θ.

$$2x = 30°, 180° - 30°, 360° + 30°, 360° + 180° - 30°$$
$$= 30°, 150°, 390°, 510°$$
$$\therefore x = 15°, 75°, 195°, 255°$$

You can solve trigonometric equations involving **radians**. You can recognise these because the domain is in radians.

Trigonometric equations

EXAMPLE 18

Solve each equation for $[0, 2\pi]$.

a $\cos x = 0.34$

b $\sin \alpha = -\frac{1}{\sqrt{2}}$

c $\sin^2 x - \sin x = 2$

The domain $[0, 2\pi]$ tells us that the solutions will be in **radians**. Make sure that your calculator is in radians mode here.

Solution

a $\cos x > 0$ in the 1st and 4th quadrants.

$$\cos x = 0.34$$
$$x \approx 1.224, 2\pi - 1.224$$
$$= 1.224, 5.059$$

b $\sin \alpha$ is negative in the 3rd and 4th quadrants.

$$\sin \alpha = -\frac{1}{\sqrt{2}}$$
$$\alpha = \pi + \frac{\pi}{4}, 2\pi - \frac{\pi}{4}$$
$$= \frac{5\pi}{4}, \frac{7\pi}{4}$$

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c $\sin^2 x - \sin x = 2$

$\sin^2 x - \sin x - 2 = 0$ This is a quadratic equation.

$(\sin x - 2)(\sin x + 1) = 0$

$\sin x = 2 \qquad \sin x = -1$

$\sin x = 2$ has no solutions since $-1 \le \sin x \le 1$

$\sin x = -1$ has solution $x = \dfrac{3\pi}{2}$

$x = \dfrac{3\pi}{2}$

Exercise 9.05 Trigonometric equations

1 Solve each equation for [0°, 360°]:

a $\sin\theta = 0.35$ **b** $\cos\theta = -\dfrac{1}{2}$ **c** $\tan\theta = -1$

d $\sin\theta = \dfrac{\sqrt{3}}{2}$ **e** $\tan\theta = -\dfrac{1}{\sqrt{3}}$ **f** $2\cos\theta = \sqrt{3}$

g $\tan 2\theta = \sqrt{3}$ **h** $2\cos 2\theta - 1 = 0$ **i** $2\sin 3\theta = -1$

j $\tan^2 3\theta = 1$ **k** $\sin^2 x = 1$ **l** $2\cos^2 x - \cos x = 0$

2 Solve for $0° \le x \le 360°$:

a $\cos x = 1$ **b** $\sin x + 1 = 0$ **c** $\cos^2 x = 1$

d $\sin x = 1$ **e** $\tan x = 0$ **f** $\sin^2 x + \sin x = 0$

g $\cos^2 x - \cos x = 0$ **h** $\tan^2 x = \tan x$ **i** $\tan^2 x = 3$

3 Solve for $[0, 2\pi]$:

a $\sin x = 0$ **b** $\tan 2x = 0$ **c** $\sin x = -1$

d $\cos x - 1 = 0$ **e** $\cos x = -1$

4 Solve for [−180°, 180°]:

a $\cos\theta = 0.187$ **b** $\sin\theta = \dfrac{1}{2}$ **c** $\tan\theta = 1$

d $\sin\theta = -\dfrac{\sqrt{3}}{2}$ **e** $\tan\theta = -\dfrac{1}{\sqrt{3}}$ **f** $3\tan^2\theta = 1$

g $\tan\theta + 1 = 0$ **h** $\tan 2\theta = 1$

5 Solve for $0 \le x \le 2\pi$:

a $\cos x = \dfrac{1}{2}$ **b** $\sin x = -\dfrac{1}{\sqrt{2}}$ **c** $\tan x = 1$

d $\tan x = \sqrt{3}$ **e** $\cos x = -\dfrac{\sqrt{3}}{2}$

6 Solve for $-\pi \le x \le \pi$:

a $2 \sin x = \sqrt{3}$ **b** $2 \cos x = 0$ **c** $3 \tan^2 x = 1$

7 Solve $2 \cos x = -1$ in the domain $[-2\pi, 2\pi]$.

8 Solve for $[0, 2\pi]$:

a $\tan^2 x + \tan x = 0$ **b** $\sin^2 x - \sin x = 0$ **c** $2 \cos^2 x - \cos x - 1 = 0$

d $4 \sin^2 x = 1$ **e** $\tan x \cos x + \tan x = 0$ **f** $\sin^2 x + 2 \cos x - 2 = 0$

9.06 Applications of trigonometric functions

Applications of trigonometric functions

WS Applying trigonometric functions

Trigonometric graphs can model real-life situations.

EXAMPLE 19

This table shows the average maximum monthly temperatures in Sydney.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
°C	26.1	26.1	25.1	22.8	19.8	17.4	16.8	18.0	20.1	22.2	23.9	25.6

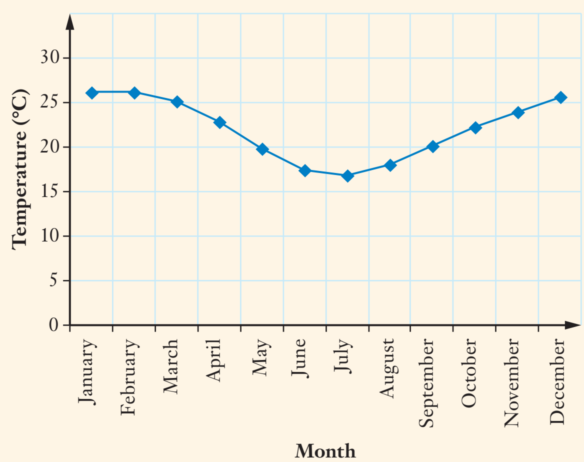
a Draw a graph of this data.

b Is it periodic? Would you expect it to be periodic?

c What is the period and amplitude?

Solution

a



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b The graph looks like it is periodic, and we would expect it to be, since the temperature varies with the seasons and these repeat every 12 months. It goes up and down, and reaches a highest value in summer and a lowest value in winter.

c This curve is approximately a cosine curve with a period of 12 months.

The highest maximum temperature is around 26° and the lowest maximum temperature is around 18°, so the centre of the graph is $\frac{26° + 18°}{2} = 22°$. So the amplitude is $26 - 22$ (or $22 - 18$) $= 4$.

DID YOU KNOW?

Waves

The sine and cosine curves are used in many applications including the study of waves. There are many different types of waves, including water, light and sound waves. Oscilloscopes display patterns of electrical waves on the screen of a cathode-ray tube.

Shutterstock.com/Oskari Porkka

Simple harmonic motion (such as the movement of a pendulum) is a wave-like or oscillatory motion when graphed against time. In 1581, when he was 17 years old, the Italian scientist Galileo noticed a lamp swinging backwards and forwards in Pisa cathedral. He found that the lamp took the same time to swing to and fro, no matter how much weight it had on it. This led him to discover the pendulum.

Galileo also invented the telescope. Find out more about his life and work.

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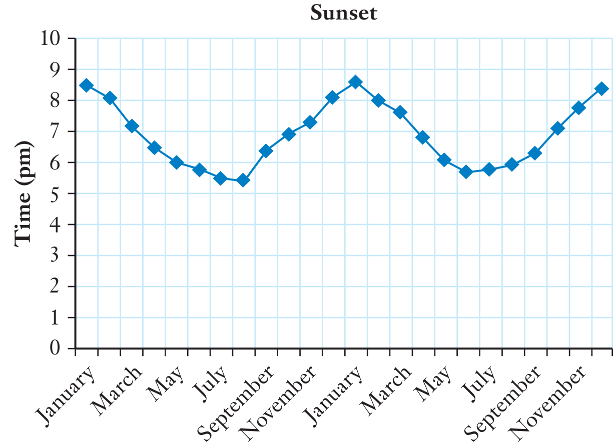
Exercise 9.06 Applications of trigonometric functions

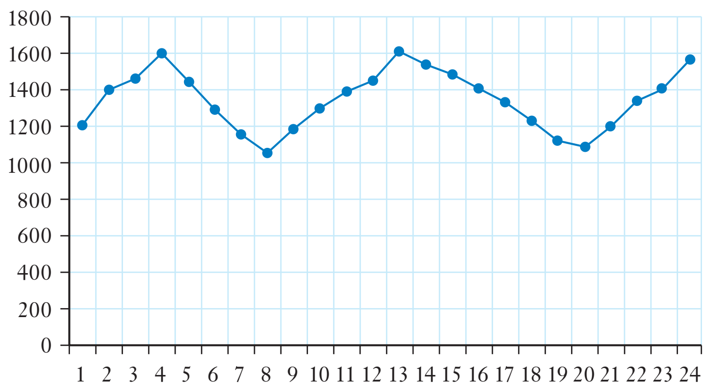
1 This graph shows the time of sunset in a city over a period of 2 years.



a Find the approximate period and amplitude of the graph.

b At approximately what time would you expect the Sun to set in July?

2 The graph shows the incidence of crimes committed over 24 years in Gotham City.



a Approximately how many crimes were committed in the 10th year?

b What was:

i the highest number of crimes? **ii** the lowest number of crimes?

c Find the approximate amplitude and the period of the graph.

3 This table shows the tides (in metres) at a jetty measured 4 times each day for 3 days.

Day	Friday				Saturday				Sunday			
Time	6:20 a.m.	11:55 a.m.	6:15 p.m.	11:48 p.m.	6:20 a.m.	11:55 a.m.	6:15 p.m.	11:48 p.m.	6:20 a.m.	11:55 a.m.	6:15 p.m.	11:48 p.m.
Tide (m)	3.2	1.1	3.4	1.3	3.2	1.2	3.5	1.1	3.4	1.2	3.5	1.3

a Draw a graph showing the tides.

b Find the period and amplitude.

c Estimate the height of the tide at around 8 a.m. on Friday.

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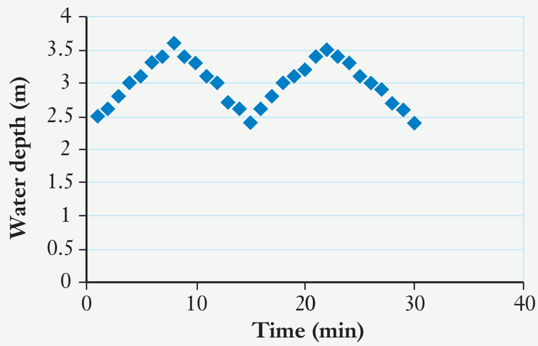
9. TEST YOURSELF

For Questions **1** to **4** select the correct answer **A, B, C** or **D**.

Practice quiz

1 This graph shows the water depth in metres as a lock opens and closes over time.



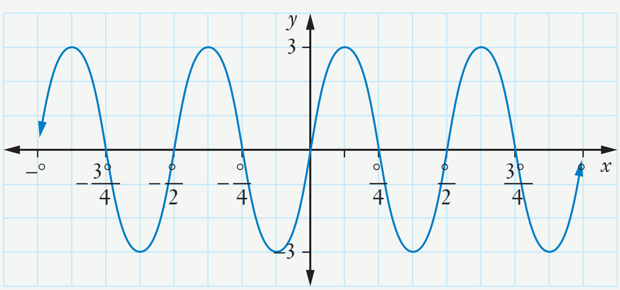
The approximate period and amplitude of the graph are:

A Amplitude 1, period 15 min
B Amplitude 0.5, period 7.5 min
C Amplitude 1, period 7.5 min
D Amplitude 0.5, period 15 min

2 The exact value of $\cos \dfrac{2\pi}{3}$ is:

A $\dfrac{1}{2}$ **B** $-\dfrac{\sqrt{3}}{2}$ **C** $\dfrac{\sqrt{3}}{2}$ **D** $-\dfrac{1}{2}$

3 The equation of the graph below is:



A $y = 3 \cos 4x$ **B** $y = 3 \sin 4x$ **C** $y = 4 \sin 3x$ **D** $y = 4 \cos 3x$

4 $\text{cosec}^2 x$ is equal to:

A $\cot^2 x - 1$ **B** $1 - \cos^2 x$ **C** $1 + \cot^2 x$ **D** $\tan^2 x + 1$

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5 Find the exact value of:

a $\cos 315°$ **b** $\sin(-60°)$ **c** $\tan 120°$

6 Solve for $0° \le x \le 360°$:

a $\sin x = \frac{\sqrt{3}}{2}$ **b** $\tan x = 1$ **c** $2\cos x + 1 = 0$

d $\sin^2 x = \frac{3}{4}$ **e** $\tan 2x = \frac{1}{\sqrt{3}}$

7 Solve for $0 \le x \le 2\pi$:

a $\tan x = -1$ **b** $2\sin x = 1$ **c** $\tan^2 x = 3$

d $\cos x = 1$ **e** $\sin x = -1$

8 For $0 \le x \le 2\pi$, sketch the graph of:

a $y = 3\cos 2x$ **b** $y = 7\sin\frac{x}{2}$

9 If $\sin x = -\frac{12}{13}$ and $\cos x > 0$, evaluate $\cos x$ and $\tan x$.

10 Simplify:

a $\cos(180° + \theta)$ **b** $\tan(-\theta)$

c $\sin(\pi - \theta)$ **d** $\tan x \cos x$

e $\sqrt{4 - 4\sin^2 A}$ **f** $\cos(90 - x)°$

g $\cot\beta \tan\beta$

11 Find the exact value of:

a $\sin\frac{5\pi}{4}$ **b** $\cos\frac{5\pi}{6}$ **c** $\tan\frac{4\pi}{3}$

12 Prove that $\frac{2\cos^2\theta}{1 - \sin\theta} = 2 + 2\sin\theta$.

13 Find the value of b if $\sin b = \cos(2b - 30)°$.

14 Find the period, amplitude, centre and phase of $y = -2\cos\left(3x + \frac{°}{12}\right) + 5$.

15 Find the exact value of:

a $\sec\frac{\pi}{4}$ **b** $\cot\frac{\pi}{6}$

c $\operatorname{cosec}\frac{\pi}{3}$ **d** $\frac{\cos\frac{\pi}{6}}{\sin\frac{\pi}{6}}$

16 Find the domain and range of each function.

a $y = -6\sin(2x) + 5$ **b** $f(x) = 4\cos x - 3$

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9. CHALLENGE EXERCISE

1 Find the exact value of:

a sin 600°

b tan (–405°)

2 Solve $2 \cos (\theta + 10°) = -1$ for $0° \le \theta \le 360°$.

3 If $f(x) = 3 \cos \pi x$:

a find the period and amplitude of the function

b sketch the graph of $f(x)$ for $0 \le x \le 4$.

4 For $0 \le x \le 2\pi$, sketch the graph of:

a $f(x) = 2 \cos\left(x + \frac{\circ}{2}\right) + 1$

b $y = 2 - 3 \sin \left(x - \frac{\circ}{2}\right)$

c $y = \sin 2x - \sin x$

d $y = \sin x + 2 \cos 2x$

e $y = 3 \cos x - \cos 2x$

f $y = \sin x - \sin \frac{x}{2}$

5 Solve $\cos^2 x - \cos x = 0$ for $0 \le x \le 2\pi$.

6 Find the exact value of sin 120° + cos 135° as a surd with rational denominator.

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