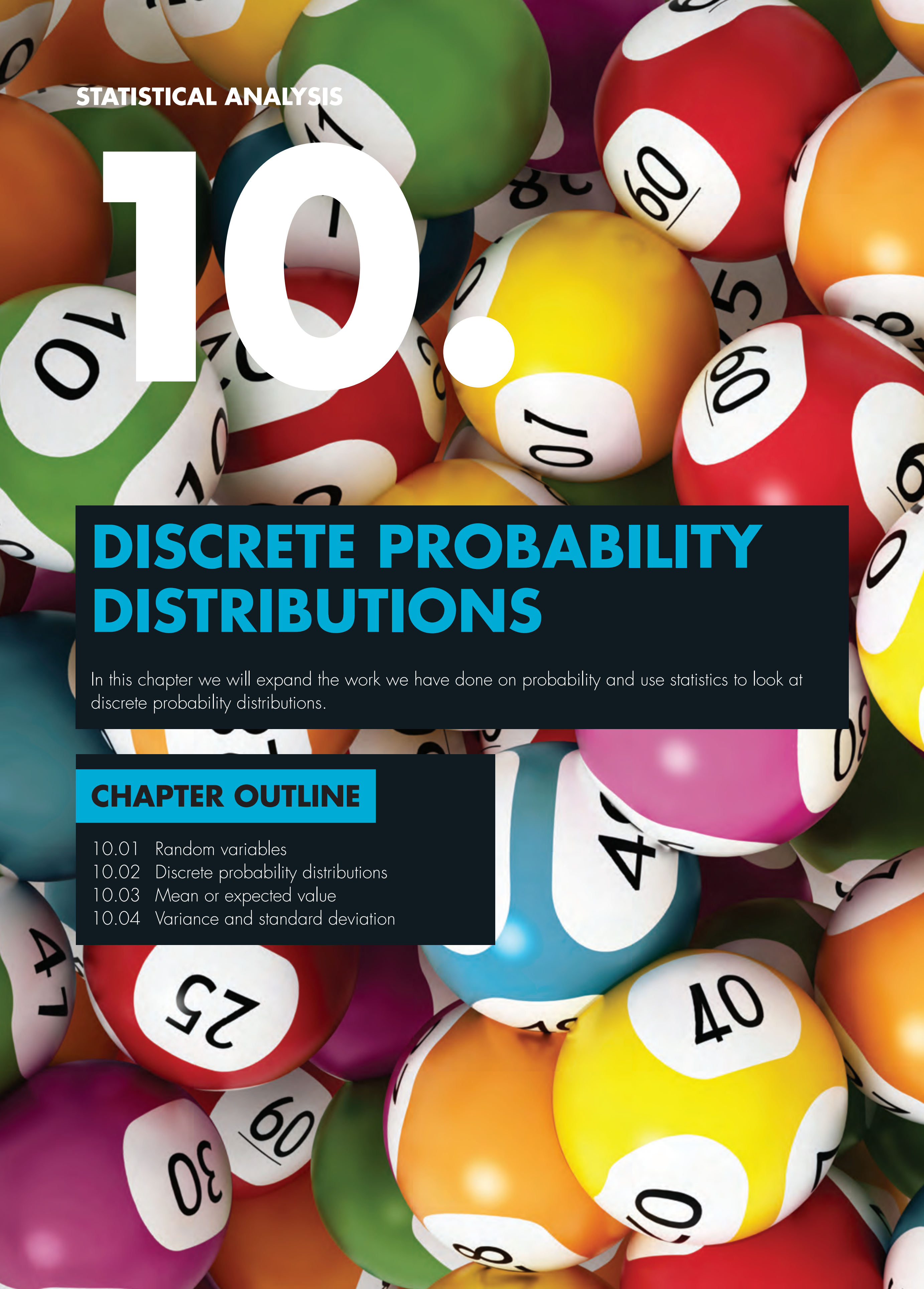

STATISTICAL ANALYSIS

10.

DISCRETE PROBABILITY DISTRIBUTIONS

In this chapter we will expand the work we have done on probability and use statistics to look at discrete probability distributions.

CHAPTER OUTLINE

IN THIS CHAPTER YOU WILL:

- understand random variables and definitions of discrete, continuous, uniform, finite and infinite variables
- recognise discrete probability distributions and their properties
- use probability distributions to solve practical problems
- find expected values, variance and standard deviations of probability distributions

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TERMINOLOGY

discrete random variable: A random variable that can take on a number of discrete values; for example, the number of children in a family

expected value: Average or mean value of a probability distribution

population: The whole data set from which a sample can be taken

probability distribution: A function that sets out all possible values of a random variable together with their probabilities

random variable: A variable whose values are based on a chance experiment; for example, the number of road accidents in an hour

standard deviation: A measure of the spread of values from the mean of a distribution; the square root of the variance

uniform probability distribution: A probability distribution in which every outcome has the same probability

variance: A measure of the spread of values from the mean of a distribution; the square of the standard deviation

10.01 Random variables

We studied probability in Chapter 7. Now we will look at **probability distributions**, which use random variables to predict and model random situations in areas such as science, economics and medicine.

A **random variable** is a variable that can take on different values depending on the outcome of a random process, such as an experiment. Random variables can be **discrete** or **continuous**. Discrete variables such as goals scored or number of children take on specific finite values, while continuous variables such as length or temperature are measured along a continuous scale.

Discrete random variables

A **discrete random variable** is a variable whose values are specific and can be listed.

In this chapter we will look at **discrete random variables**. We will look at continuous random variables in Year 12.

EXAMPLE 1

Is each random variable discrete or continuous?

a The number of goals scored by a netball team

b The height of a student

c The shoe size of a Year 11 student.

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Solution

a The number of goals scored is a specific whole number so it is a discrete random variable.

b Height is measured on a continuous scale so the height of a student is a continuous random variable.

c Shoe sizes are specific values that can be listed so it is a discrete random variable.

We use a capital letter such as X for a random variable and a lower-case letter such as x for the values of X.

EXAMPLE 2

Find the set of possible values for each discrete random variable:

a The number rolled on a die

b The number of girls in a family of 3 children

c The number of heads when tossing a coin 8 times.

Solution

a Any number from 1 to 6 can be rolled on a die.

So $X = \{1, 2, 3, 4, 5, 6\}$

b It is possible to have no girls, 1 girl, 2 girls or 3 girls.

So $X = \{0, 1, 2, 3\}$

c The coin could come up heads 0, 1, 2, … 8 times.

So $X = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$

Exercise 10.01 Random variables

1 For each random variable, state whether it is discrete or continuous:

a A film critic's rating of a film, from 0 to 4 stars

b The speed of a car

c The sum rolled on a pair of dice

d The winning ticket number drawn from a raffle

e The weight of parcels at a post office

f The size of jeans in a shop

g The temperature of a metal as it cools

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h The amount of water in different types of fruit drink
i The number of cars passing the school over a 10-minute period
j The number of cities in each country in Europe
k The number of heads when tossing a coin 50 times
l The number of correct answers in a 10-question test.

2 Write the set of possible values for each discrete random variable:
a Number of daughters in a one-child family
b Number of 6s on 10 rolls of a die
c Number of people aged over 50 in a group of 20 people
d The number of days it rains in March
e The sum of the 2 numbers rolled on a pair of dice.

Discrete probability distributions 1

Discrete probability distributions 2

Probability density functions

10.02 Discrete probability distributions

Discrete probability distribution

A **discrete probability distribution** lists the probability for each value of a discrete random variable.

A discrete probability distribution can be displayed in a table or graph, or represented by an equation or set of ordered pairs. It is also called a **discrete probability function**.

We can write a probability function that uses X as the random variable as $P(X = x)$ or $p(x)$.

EXAMPLE 3

a In a random experiment, a die was rolled and the results recorded in the table below.

Number	Frequency
1	18
2	23
3	17
4	28
5	12
6	22

i How many times was the die rolled?

ii Draw up a probability distribution for this experiment.

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b Write the probability function of rolling a die as a set of ordered pairs $(x, P(X = x))$.

c A probability function for discrete random variable X is given by:

$$P(X=x)=\begin{cases}\frac{1}{16}(4-x) & \text{for } x = 0, 2\\ \frac{1}{8}(x-1) & \text{for } x = 3, 4\\ 0 & \text{for any other } x \text{ value}\end{cases}$$

i Complete a discrete probability distribution table.

ii Find the sum of all probabilities.

iii Evaluate $P(X = \text{odd})$.

iv Evaluate $P(X \leq 3)$.

Solution

a **i** Adding the frequencies, the die was rolled 120 times.

ii For the probability of each outcome, we use **relative frequencies** as we did in Chapter 7.

x	1	2	3	4	5	6
$P(X = x)$	$\frac{18}{120}=\frac{3}{20}$	$\frac{23}{120}$	$\frac{17}{120}$	$\frac{28}{120}=\frac{7}{30}$	$\frac{12}{120}=\frac{1}{10}$	$\frac{22}{120}=\frac{11}{60}$

b The probability of each number being rolled on a die is $\frac{1}{6}$. So the probability function $P(X = x)$ can be written as $\left(1, \frac{1}{6}\right), \left(2, \frac{1}{6}\right), \left(3, \frac{1}{6}\right), \left(4, \frac{1}{6}\right), \left(5, \frac{1}{6}\right), \left(6, \frac{1}{6}\right)$.

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c **i** $p(0) = \frac{1}{16}(4-0)$

$= \frac{1}{4}$

$p(2) = \frac{1}{16}(4-2)$

$= \frac{1}{8}$

$p(3) = \frac{1}{8}(3-1)$

$= \frac{1}{4}$

$p(4) = \frac{1}{8}(4-1)$

$= \frac{3}{8}$

All other values of x give $p(x) = 0$. We cannot put all of these in a table. They will not make any difference to calculations anyway.

x	0	1	2	3	4
$p(x)$	$\frac{1}{4}$	0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{3}{8}$

ii $p(0) + p(1) + p(2) + p(3) + p(4) = \frac{1}{4} + 0 + \frac{1}{8} + \frac{1}{4} + \frac{3}{8}$

$= 1$

iii $P(X = \text{odd}) = p(1) + p(3)$

$= 0 + \frac{1}{4}$

$= \frac{1}{4}$

iv $P(X \le 3) = p(0) + p(1) + p(2) + p(3)$

$= \frac{1}{4} + 0 + \frac{1}{8} + \frac{1}{4}$

$= \frac{5}{8}$

Remember that all probabilities lie between 0 and 1 and their total is 1. These same rules apply to a probability distribution.

Properties of discrete probability distributions

For a discrete probability distribution:

- all possible values of X must be mutually exclusive
- the sum of all probabilities must be 1
- for each value of x, $0 \le P(X = x) \le 1$.

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EXAMPLE 4

Consider this discrete probability distribution.

x	1	2	3	4	5	6
$P(X = x)$	0.2	0.35	0.1	0.15	0.05	0.15

a Find:

i $P(X = 2)$ **ii** $P(X < 3)$

iii $P(X \geq 4)$ **iv** $P(2 \leq X < 5)$

b Show that the sum of probabilities is 1.

c Draw a histogram of the function.

Solution

a **i** From the table, $p(2) = 0.35$.

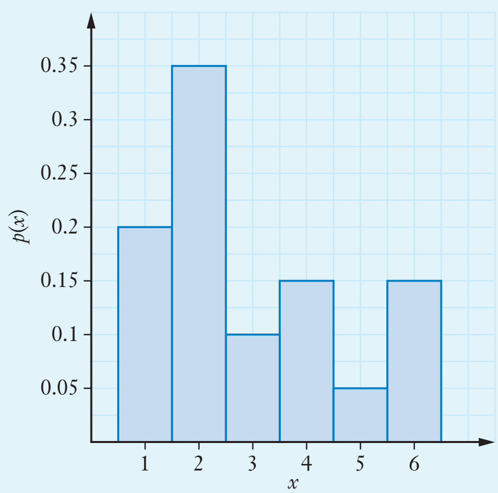
ii $$\begin{aligned} P(X < 3) &= p(1) + p(2) \\ &= 0.2 + 0.35 \\ &= 0.55 \end{aligned}$$

iii $$\begin{aligned} P(X \geq 4) &= p(4) + p(5) + p(6) \\ &= 0.15 + 0.05 + 0.15 \\ &= 0.35 \end{aligned}$$

iv $$\begin{aligned} P(2 \leq X < 5) &= p(2) + p(3) + p(4) \\ &= 0.35 + 0.1 + 0.15 \\ &= 0.6 \end{aligned}$$

b $$\begin{aligned} p(1) + p(2) + p(3) + p(4) + p(5) + p(6) &= 0.2 + 0.35 + 0.1 + 0.15 + 0.05 + 0.15 \\ &= 1 \end{aligned}$$

c A histogram is the best type of graph to draw a discrete probability distribution.



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EXAMPLE 5

a A function is given by $p(x) = \dfrac{x-1}{3}$ where $x = 1, 2, 3$. Is the function a probability distribution?

b Find the value of n for which the table below is a discrete probability distribution.

x	0	1	2	3	4
$p(x)$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{5}{12}$	$\frac{1}{12}$	n

Solution

a
$$\begin{aligned} p(1) &= \frac{1-1}{3} \\ &= 0 \end{aligned}$$
$$\begin{aligned} p(2) &= \frac{2-1}{3} \\ &= \frac{1}{3} \end{aligned}$$
$$\begin{aligned} p(3) &= \frac{3-1}{3} \\ &= \frac{2}{3} \end{aligned}$$
$$\begin{aligned} p(1) + p(2) + p(3) &= 0 + \frac{1}{3} + \frac{2}{3} \\ &= 1 \end{aligned}$$

So the function is a probability distribution.

b The sum of the probabilities must be 1.

$$\begin{aligned} \frac{1}{12} + \frac{1}{6} + \frac{5}{12} + \frac{1}{12} + n &= 1 \\ \frac{9}{12} + n &= 1 \\ n &= 1 - \frac{9}{12} \\ n &= \frac{1}{4} \end{aligned}$$

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EXAMPLE 6

A game involves scoring points for selecting a card at random from a deck of 52 playing cards. The table shows the scores awarded for different selections.

Type of card	Score
Number 2–10	2
Picture card (jack, queen or king)	5
Ace	8

a Draw a probability distribution table for random variable Y for the different scores.

b Find:

i $P(Y < 8)$ **ii** $P(Y > 2)$

Solution

a There are 9 cards numbered 2–10 in each of the 4 suits (hearts, diamonds, spades and clubs). So there are $9 \times 4 = 36$ cards that will score 2 points.

There are 3 picture cards (jack, queen and king) in each suit, so there are $3 \times 4 = 12$ cards that will score 5 points.

There are 4 aces that will score 8 points.

y	2	5	8
$P(Y = y)$	$\frac{36}{52} = \frac{9}{13}$	$\frac{12}{52} = \frac{3}{13}$	$\frac{4}{52} = \frac{1}{13}$

b **i** $P(X < 8) = p(2) + p(5)$

$$= \frac{9}{13} + \frac{3}{13}$$

$$= \frac{12}{13}$$

ii $P(X > 2) = p(5) + p(8)$

$$= \frac{3}{13} + \frac{1}{13}$$

$$= \frac{4}{13}$$

Uniform distribution

In a **uniform probability distribution**, the probability for each value is the same.

Uniform probability distribution

A uniform probability distribution occurs if random variable X has n values where

$P(X = x) = \frac{1}{n}$ for $x = 1, 2, 3, \ldots, n$

EXAMPLE 7

Which probability distribution is uniform?

a The number of heads when tossing a coin

b The number of heads when tossing 2 coins.

Solution

a When tossing a coin, we could get either 0 or 1 head.

$X = \{0, 1\}$

$p(0) = \dfrac{1}{2}$

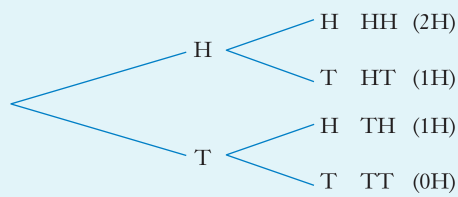
$p(1) = \dfrac{1}{2}$

Since both values have the same probability, it is a uniform distribution.

b When tossing 2 coins, the number of heads could be 0, 1 or 2.

$X = \{0, 1, 2\}$

Using a probability tree or table, we can find the probability for each outcome.



$$p(0) = P(\text{TT}) = \frac{1}{4}$$

$$p(1) = P(\text{HT}) + P(\text{TH}) = \frac{2}{4} = \frac{1}{2}$$

$$p(2) = P(\text{HH}) = \frac{1}{4}$$

The probabilities are not all the same so it is not a uniform distribution.

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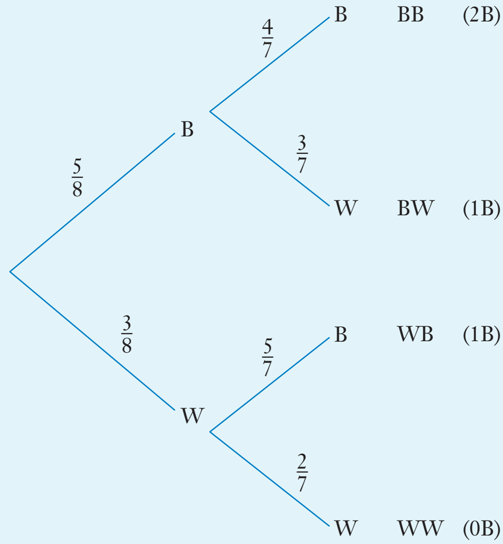
EXAMPLE 8

Draw a probability distribution table for the number of black balls that could be drawn out of a bag containing 5 black and 3 white balls when 2 balls are selected randomly without replacement.

Solution

First draw a probability tree:



$$P(0\text{B}) = P(\text{WW})$$
$$= \frac{3}{8} \times \frac{2}{7}$$
$$= \frac{6}{56}$$
$$= \frac{3}{28}$$

$$P(1\text{B}) = P(\text{BW}) + P(\text{WB})$$
$$= \frac{5}{8} \times \frac{3}{7} + \frac{3}{8} \times \frac{5}{7}$$
$$= \frac{30}{56}$$
$$= \frac{15}{28}$$

$$P(2\text{B}) = P(\text{BB})$$
$$= \frac{5}{8} \times \frac{4}{7}$$
$$= \frac{20}{56}$$
$$= \frac{5}{14}$$

x	0	1	2
$P(X = x)$	$\frac{3}{28}$	$\frac{15}{28}$	$\frac{5}{14}$

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Exercise 10.02 Discrete probability distributions

1 Draw a probability distribution table for the sum of the numbers rolled on 2 dice.

2 Write the probability distribution as a set of ordered pairs $(x, P(X = x))$ for the number of heads when tossing:

a 1 coin **b** 2 coins **c** 3 coins.

3 A survey of a sample of bags of 50 jelly beans found that they didn't all hold exactly 50. The table shows the results of the study.

Number of jelly beans	Frequency
48	8
49	9
50	21
51	9
52	6

a Draw a probability distribution table for the results.

b If a bag of jelly beans is chosen at random, find the probability that the bag contains:

i at least 50 jelly beans

ii fewer than 51 jelly beans

4 A function is given by $p(x) = \frac{x-2}{6}$ for $x = 3, 4, 5$.

a Show that the function is a probability distribution.

b Draw up a probability distribution table.

c Find:

i $P(X > 3)$ **ii** $P(X = \text{odd})$ **iii** $P(3 \le X < 5)$

5 Draw a histogram to show this discrete probability distribution.

x	0	1	2	3	4
$P(X = x)$	0.05	0.4	0.25	0.1	0.2

6 **a** Draw a probability distribution table for rolling a die.

b Is this a uniform distribution?

c Find:

i $P(X \ge 4)$ **ii** $P(X < 3)$ **iii** $P(1 < X \le 4)$

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7 For each function, state whether it is a probability distribution:

a $\left(0,\frac{1}{5}\right)$, $\left(1,\frac{2}{5}\right)$, $(2, 0)$, $\left(3,\frac{2}{5}\right)$, $\left(4,\frac{1}{5}\right)$

b

x	1	2	3	4
$P(X = x)$	$\frac{3}{10}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{10}$

c $p(x) = \dfrac{x+2}{4}$ for $x = 0, 1, 2$

8 Find k if each function is a probability distribution:

a $p(x) = k(x + 1)$ for $x = 1, 2, 3, 4$

b

x	0	1	2	3	4
$P(X = x)$	0.2	k	0.15	0.34	0.12

c $(1, k), \left(2,\frac{1}{10}\right), (3, 0), \left(4,\frac{1}{5}\right), \left(5,\frac{3}{10}\right), \left(6,\frac{2}{5}\right)$

9 The probability function for the random variable X is given by $p(x) = \dfrac{kx^2}{x+5}$ for $x = 1, 2, 3, 4$.

a Construct a probability distribution table for the function.

b Find the value of k.

10 In a game, each player rolls 2 dice. The game pays \$1 if one of the numbers is a 6, \$3 for double 6 and \$2 for any other double. There is no payout for other results.

a Draw a probability distribution table for the game payout Y.

b Find the probability of winning:

i \$3 **ii** at least \$2 **iii** less than \$3.

11 Given the probability function below, evaluate p.

x	3	4	5	6
$P(X = x)$	$2p$	$3p$	$5p$	p

12 Simon plays a game where he selects a card at random from 100 cards numbered 1 to 100. He wins \$1 for selecting a number less than 20, \$2 for a number greater than 90, \$3 for any number from 61 and 69 (inclusive) and \$5 for any number from 41 to 50 (inclusive).

a Create a probability distribution table for the random variable X for the prize values.

b Find the probability of winning more than \$2.

c Find the probability of winning less than \$5.

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13 The table below shows the probability function for random variable X.

x	5	6	7	8	9	10
$P(X = x)$	$\frac{3}{14}$	$\frac{2}{7}$	$\frac{1}{14}$	$\frac{1}{14}$	$\frac{3}{14}$	$\frac{1}{7}$

Find:

a $P(X = 6)$ **b** $P(X = \text{even})$ **c** $P(X > 8)$ **d** $P(X \le 7)$

e $P(6 < X < 9)$ **f** $P(7 \le X < 10)$ **g** $P(6 \le X \le 9)$

14 The table below shows the probability function for random variable Q.

q	0	2	4	6	8	10
$P(Q = q)$	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{3}{16}$	$\frac{3}{16}$

Find:

a $p(8)$ **b** $P(Q \ge 4)$ **c** $P(2 < Q \le 6)$

d $P(4 \le Q \le 10)$ **e** $P(0 \le Q < 4)$ **f** $P(2 \le Q \le 8)$

15 A company makes washing machines. On average, there are 3 faulty machines made for every 1000 machines. Two washing machines are selected at random for a quality control inspection.

a Draw a probability distribution table for the number of these machines that could be faulty.

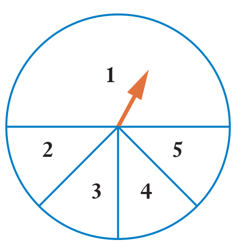
b Find the probability that:

i one will be faulty **ii** at least one will be faulty.

16 A bag contains 7 red, 6 white and 8 blue balls. Create a probability function table for the number of white balls selected when drawing 2 balls at random from the bag:

a with replacement **b** without replacement.

17 The spinner below has the numbers 1–5 distributed as shown:



a What is the probability that the arrow points to the number 3 when it is spun?

b Is the probability distribution of the spun numbers uniform?

c Draw a table showing the probability distribution for spinning the numbers.

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18 The probability of a traffic light showing green as a car approaches it is 12%. Draw a probability distribution table for the number of green traffic lights on approach when a car passes through 3 traffic lights.

Shutterstock.com/Vacclav

19 There is a 51% chance of giving birth to a boy. If a family has 4 children, construct a probability function to show the number of boys in the family.

20 A raffle has 2 prizes with 100 tickets sold altogether. Iris buys 5 tickets.

a Draw a probability distribution table to show the number of prizes Iris could win in the raffle.

b Find the probability that Iris wins at least one prize.

10.03 Mean or expected value

WS Expected value

The **expected value** $E(X)$ of a probability distribution measures the centre of the distribution. It is the same as finding the **mean**, or average, which has symbol μ. It is the expected value of the random variable.

We use $\bar{x}$ for the mean of a **sample** and μ for the mean of a **population**. For probability distributions, we use the population mean, μ. The sample mean, $\bar{x}$, is an estimate of μ, and as the sample size increases, the sample represents the population better and the value of $\bar{x}$ approaches μ.

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EXAMPLE 9

This table shows Harrison's diving scores (out of 10) over one year.

Score	Frequency
5	7
6	9
7	8
8	3
9	2
10	1

a Copy the table and add 2 columns to calculate the relative frequency for each score and the product of each score and its relative frequency.

b Calculate, correct to 2 decimal places, the sum of the last column (products of scores and their relative frequencies) to find Harrison's expected value (average score) for the year.

Solution

a Adding frequencies gives a total of 30.

Score	Frequency	Relative frequency	Score × relative frequency
5	7	$\frac{7}{30}$	$5 \times \frac{7}{30} = \frac{35}{30}$
6	9	$\frac{9}{30}$	$6 \times \frac{9}{30} = \frac{54}{30}$
7	8	$\frac{8}{30}$	$7 \times \frac{8}{30} = \frac{56}{30}$
8	3	$\frac{3}{30}$	$8 \times \frac{3}{30} = \frac{24}{30}$
9	2	$\frac{2}{30}$	$9 \times \frac{2}{30} = \frac{18}{30}$
10	1	$\frac{1}{30}$	$10 \times \frac{1}{30} = \frac{10}{30}$

b
$$\begin{aligned}\text{Expected value} &= \frac{35}{30} + \frac{54}{30} + \frac{56}{30} + \frac{24}{30} + \frac{18}{30} + \frac{10}{30}\\ &= \frac{197}{30}\\ &= 6.5666\ldots\\ &= 6.57\end{aligned}$$

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INVESTIGATION

MEAN

Find the mean in Example 9 using the formula $\overline{x} = \frac{\sum fx}{\sum f}$. Can you see why the sum of scores multiplied by relative frequencies also gives this mean?

Expected value

$$E(X) = \mu = \sum xp(x)$$

The symbol Σ means 'the sum of'. It is the Greek capital letter 'sigma'.

$\sum xp(x)$ is the sum of the products of x times $p(x)$.

Proof

$$\mu = \frac{\sum fx}{\sum f}$$

$$= \frac{\sum fx}{n} \text{ where } n \text{ is the sum of frequencies}$$

$$= \sum x\frac{f}{n}$$

$$= \sum xp(x)$$

EXAMPLE 10

a Find the expected value of this discrete probability distribution.

x	1	2	3	4	5
$P(X = x)$	0.1	0.3	0.2	0.1	0.3

b In a game of chance, Bethany tosses 2 coins. She wins \$10 for 2 heads, \$5 for 2 tails and nothing for a head and a tail.

- **i** Find the expected value of this game.
- **ii** If the game costs \$5 to play, would Bethany expect to win or lose money in the long term?

Solution

a $E(X) = \sum xp(x)$
$= 1 \times 0.1 + 2 \times 0.3 + 3 \times 0.2 + 4 \times 0.1 + 5 \times 0.3$
$= 3.2$

b **i** We can make $X = \{0, 1, 2\}$ where 0, 1 and 2 is the number of heads.

x	0	1	2
$P(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

The game pays different amounts of money for different outcomes.

We can use Y as the random variable for the payout of money in dollars.

$Y = \{0, 5, 10\}$

1 head earns \$0, 0 heads earn \$5, 2 heads earn \$10.

y	\$0	\$5	\$10
$P(Y = y)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

$$\begin{aligned} E(Y) &= \sum yp(y) \\ &= \$0 \times \frac{1}{2} + \$5 \times \frac{1}{4} + \$10 \times \frac{1}{4} \\ &= \$3.75 \end{aligned}$$

ii Since the game costs \$5 to play and the expected outcome is \$3.75, Bethany would expect to lose money in the long term.

You can find the expected value on your calculator using the statistics mode, in the same way you would find the mean of data presented in a frequency table.

EXAMPLE 11

Find the expected value of this discrete probability distribution.

x	0	2	4	6	8
$P(X = x)$	0.14	0.13	0.25	0.36	0.12

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Solution

Operation	Casio scientific	Sharp scientific
Place your calculator in statistical mode:	MODE STAT 1-VAR SHIFT MODE scroll down to STAT Frequency? ON	MODE STAT =
Clear the statistical memory:	SHIFT 1 Edit, Del-A	2ndF DEL
Enter data:	SHIFT 1 Data to get table. 0 = 2 = etc. to enter in x column. 0.14 = 0.13 = etc. to enter in FREQ column. AC to leave table.	0 2ndF STO 0.14 M+ 2 2ndF STO 0.13 M+ etc.
Calculate mean: ($\overline{x} = 4.38$)	SHIFT VAR $\overline{x}$ =	RCL $\overline{x}$
Check the number of scores: ($n = 1$)	SHIFT VAR n =	RCL n
Change back to normal mode:	MODE COMP	MODE 0

Expected value $E(X) = 4.38$.

You can solve problems using expected values.

EXAMPLE 12

a The probability of selling a red car, based on previous experience, is 35%. Find the expected number of red cars sold in one week if a dealer sells 2 cars.

b For the probability function below, evaluate a and b given that $E(X) = 2$.

x	1	2	3	4
$P(X = x)$	a	b	0.2	0.1

Solution

a $P(\text{R}) = 35\%$

$= 0.35$

So $P(\text{not R}) = P\left(\bar{\text{R}}\right)$

$= 1 - 0.35$

$= 0.65$

$P(X = 0) = P(\bar{\text{R}}\bar{\text{R}})$

$= 0.65 \times 0.65$

$= 0.4225$

$P(X = 1) = P(\bar{\text{R}}\text{R}) + P(\text{R}\bar{\text{R}})$

$= 0.65 \times 0.35 + 0.35 \times 0.65$

$= 0.455$

$P(X = 2) = P(\text{RR})$

$= 0.35 \times 0.35$

$= 0.1225$

x	0	1	2
$P(X = x)$	0.4225	0.455	0.1225

$E(X) = \sum xp(x)$

$= 0 \times 0.4225 + 1 \times 0.455 + 2 \times 0.1225$

$= 0.7$

So it is expected that when 2 cars are sold, 0.7 of them will be red.

Rounded to the nearest whole number, we could expect around 1 car to be red.

Note: This answer would be more meaningful for a much larger number of sales!

b $E(X) = \sum xp(x)$

$2 = 1 \times a + 2 \times b + 3 \times 0.2 + 4 \times 0.1$

$2 = a + 2b + 1$

$a + 2b = 1 \qquad [1]$

Since the function is a probability distribution:

$a + b + 0.2 + 0.1 = 1$

$a + b + 0.3 = 1$

$a + b = 0.7 \qquad [2]$

$[1] - [2]: b = 0.3$

Substitute into [2]:

$a + 0.3 = 0.7$

$a = 0.4$

So $a = 0.4$, $b = 0.3$.

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Exercise 10.03 Mean or expected value

1 Find the expected value of each probability distribution.

a $\left(0, \frac{1}{4}\right), \left(1, \frac{1}{2}\right), \left(2, \frac{1}{4}\right)$

b

x	1	2	3	4	5
$P(X = x)$	0.31	0.16	0.15	0.2	0.18

c

x	1	2	3	4	5
$P(X = x)$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{3}{16}$

d $p(x) = \frac{x+1}{8}$ for $x = 0, 1, 4$

e $$p(x) = \begin{cases} \frac{x}{2} & \text{for } x = 1 \\ \frac{x}{8} & \text{for } x = 2 \\ \frac{x-3}{4} & \text{for } x = 4 \\ 0 & \text{for all other values of } x \end{cases}$$

2 For each question:

i evaluate k **ii** find the mean of the probability distribution.

a $(1, k), \left(2, \frac{1}{5}\right), \left(3, \frac{3}{10}\right), \left(4, \frac{2}{5}\right)$

b $p(x) = k(x + 3)$ for $x = 0, 1, 2$

c

x	1	2	3	4	5	6
$P(X = x)$	0.1	0.02	0.17	0.24	k	0.32

3 Find the expected value of each probability distribution:

a The number of heads when tossing 2 coins

b The sum of the 2 numbers rolled on a pair of dice

c The number of girls in a 3-child family

d The number of faulty cars when testing 3 cars if 1 in every 1000 cars is faulty

e The number of red counters when 2 counters are selected at random from a bag containing 7 red and 12 white counters:

i with replacement

ii without replacement.

4 The expected value $E(X) = 6.35$ for this probability function. Find p and q.

x	3	7	8	9
$p(x)$	p	0.25	0.35	q

5 The mean of the probability distribution below is $3\frac{3}{4}$. Evaluate a and b.

$(1, a), \left(2, \frac{1}{8}\right), (3, b), \left(4, \frac{1}{4}\right), \left(5, \frac{3}{8}\right)$

6 A uniform discrete random variable X has values $x = 1, 2, 3, 4$.

a Draw up a probability distribution table for X.

b Find $E(X)$.

7 Find the expected number of heads when tossing 3 coins.

8 A bag contains 8 white and 3 yellow marbles. If 3 marbles are selected at random, find the mean number of white marbles:

a with replacement

b without replacement.

9 In a game, 2 dice are rolled and the difference between the 2 numbers is calculated. A player wins \$1 if the difference is 3, \$2 if it is 4 and \$3 if it is 5.

a Draw a probability distribution table for the winning values.

b Find the expected value.

c It costs \$1 for a player to roll the dice. How much would the player be expected to win or lose?

10 Staff at a call centre must make at least 1 phone sale every hour. The probability that Yasmin will make a sale on a phone call is $\frac{2}{5}$. She makes 4 phone calls in an hour.

a Draw a probability distribution for the number of sales Yasmin makes.

b Find the expected value.

c Will Yasmin make at least one phone sale in an hour?

11 A game uses a spinner with the numbers 1 to 12 equally spread around it. A player wins \$3 for spinning a number greater than 10, \$2 for a number less than 4 and loses \$1 for any other number. How much money would a player be expected to win or lose?

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10.04 Variance and standard deviation

Variance and standard deviation measure the spread of data in a distribution by finding the difference of each value from the mean. Variance is the square of the standard deviation.

The variance, σ^2, involves the average of the squared differences of each value from the mean.

σ is the Greek lower-case letter 'sigma'.

Variance and standard deviation

Variance and standard deviation of a discrete random variable

Variance, σ^2

$$\begin{aligned} Var(X) &= \Sigma(x-\mu)^2 p(x) \\ &= E[(X-\mu)^2] \end{aligned}$$

Proof

$$\begin{aligned} \sigma^2 &= \frac{\Sigma f(x-\mu)^2}{\Sigma f} \\ &= \frac{\Sigma f(x-\mu)^2}{n} \qquad \text{where } n = \Sigma f \text{ is the sum of frequencies} \\ &= \Sigma(x-\mu)^2 \times \frac{f}{n} \\ &= \Sigma(x-\mu)^2 p(x) \\ &= E[(X-\mu)^2] \end{aligned}$$

Standard deviation is the square root of variance.

Standard deviation, σ

$$\begin{aligned} \sigma &= \sqrt{\Sigma(x-\mu)^2\, p(x)} \\ &= \sqrt{E[(X-\mu)^2]} \end{aligned}$$

We use s for the standard deviation of a **sample** and σ (the lower-case Greek letter sigma) for the standard deviation of a population. For probability distributions, we use the population standard deviation, σ. The sample standard deviation, s, is an estimate of σ, and as the sample size increases, the sample represents the population better and the value of s approaches σ.

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EXAMPLE 13

Find the variance and standard deviation of this probability distribution.

x	1	2	3	4
$P(X = x)$	0.16	0.32	0.42	0.1

Solution

$$E(X) = \sum xp(x)$$
$$= 1 \times 0.16 + 2 \times 0.32 + 3 \times 0.42 + 4 \times 0.1$$
$$= 2.46$$
$$Var(X) = \sum(x - \mu)^2 p(x)$$
$$= (1 - 2.46)^2 0.16 + (2 - 2.46)^2 0.32 + (3 - 2.46)^2 0.42 + (4 - 2.46)^2\, 0.1$$
$$= 0.7684$$

Standard deviation:

$$\sigma = \sqrt{0.7684}$$
$$= 0.8765\ldots$$
$$\approx 0.8766$$

The formula for variance is a little tedious since we subtract the mean from every value. There is a simpler formula for variance.

Calculation formulas for variance and standard deviation

$$Var(X) = \sum[x^2 p(x)] - \mu^2$$
$$= E(X^2) - \mu^2$$
$$\sigma = \sqrt{Var(X)}$$

Proof

$\sigma^2 = \sum(x - \mu)^2 p(x)$

$= \sum[x^2 p(x) - 2\mu x p(x) + \mu^2 p(x)]$ expanding $(x - \mu)^2$

$= \sum x^2 p(x) - \sum 2\mu x p(x) + \sum \mu^2 p(x)$ taking separate sums of each part

$= \sum x^2 p(x) - 2\mu \sum x p(x) + \mu^2 \sum p(x)$ since μ is a constant

$= \sum x^2 p(x) - 2\mu \times \mu + \mu^2 \times 1$ since $\sum xp(x) = \mu$ and $\sum p(x) = 1$ is a constant

$= \sum[x^2 p(x)] - \mu^2$

$= E(X^2) - \mu^2$

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If we use the same probability distribution as Example 13, we can see that this formula gives us the same result.

EXAMPLE 14

Use the simpler calculation formulas to find the variance and standard deviation of this probability distribution.

x	1	2	3	4
$P(X = x)$	0.16	0.32	0.42	0.1

Solution

$$\begin{aligned}\mu = E(X) &= \sum xp(x)\\ &= 1 \times 0.16 + 2 \times 0.32 + 3 \times 0.42 + 4 \times 0.1\\ &= 2.46\end{aligned}$$

$$\begin{aligned}Var(X) &= \sum[x^2 p(x)] - \mu^2\\ &= (1)^2\ 0.16 + (2)^2\ 0.32 + (3)^2\ 0.42 + (4)^2\ 0.1 - 2.46^2\\ &= 0.7684\end{aligned}$$

Standard deviation:

$$\begin{aligned}\sigma &= \sqrt{0.7684}\\ &= 0.8765\ldots\\ &\approx 0.8766\end{aligned}$$

You can use a calculator to work out the variance and standard deviation.

EXAMPLE 15

Find the expected value, standard deviation and variance of this probability distribution.

x	1	2	3	4	5
$P(X = x)$	0.1	0.25	0.2	0.35	0.1

Solution

Operation	Casio scientific	Sharp scientific
Place your calculator in statistical mode:	MODE STAT 1-VAR SHIFT MODE scroll down to STAT Frequency? ON	MODE STAT =
Clear the statistical memory:	SHIFT 1 Edit, Del-A	2ndF DEL
Enter data:	SHIFT 1 Data to get table. 1 = 2 = etc. to enter in x column. 0.1 = 0.25 = etc to enter in FREQ column. AC to leave table.	1 2ndF STO 0.1 M+ 2 2ndF STO 0.25 M+ etc.
Calculate mean: ($\bar{x} = 3.1$)	SHIFT Var $\bar{x}$ =	RCL $\bar{x}$
Calculate the standard deviation: ($\sigma_x = 1.1789\ldots$)	SHIFT Var σ_x =	RCL σ_x
Change back to normal mode:	MODE COMP	MODE 0

Mean $\mu = 3.1$

Standard deviation $\sigma \approx 1.18$

Variance $\sigma^2 = 1.1789\ldots^2$

≈ 1.39

Exercise 10.04 Variance and standard deviation

In this exercise, round answers to 2 decimal places where necessary.

1 For each probability distribution, find:

i the standard deviation **ii** the variance.

a

x	1	2	3	4	5	6
$P(X = x)$	0.17	0.24	0.12	0.13	0.23	0.11

b

x	0	1	2	3
$P(X = x)$	$\frac{1}{7}$	$\frac{3}{7}$	$\frac{1}{14}$	$\frac{5}{14}$

c $\left(1, \frac{3}{8}\right), \left(2, \frac{1}{4}\right), \left(3, \frac{1}{8}\right), \left(4, \frac{1}{16}\right), \left(5, \frac{3}{16}\right)$

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2 Find the mean, variance and standard deviation of each probability function.

a

x	1	4	7	9	10
$p(x)$	0.09	0.18	0.26	0.32	0.15

b $P(x) = \frac{x+1}{9}$ for $x = 0, 2, 4$

3 Evaluate n and find the expected value and variance for this probability distribution.

x	1	2	3	4	5
$p(x)$	$\frac{2}{5}$	$\frac{1}{10}$	$\frac{3}{20}$	$\frac{1}{20}$	n

4 For the probability distribution below with $E(X) = 3.32$, find:

x	1	2	3	4	5
$p(x)$	a	b	0.17	0.2	0.3

a the values of a and b

b the variance

c the standard deviation.

5 For each probability distribution find:

i the expected value

ii the standard deviation

iii the variance.

a The number of tails when tossing 3 coins.

b The number of blue marbles when 2 marbles are selected randomly from a bag containing 10 blue and 12 white marbles.

6 A uniform discrete random variable X has values $x = 1, 2, 3, 4, 5$. Find:

a the mean

b the standard deviation

c the variance.

7 a Create a probability distribution table for the number of 6s rolled on a pair of dice.

b Find the mean, variance and standard deviation of this function.

8 The probability of selecting a black jelly bean at random from a packet is 4%. If 2 jelly beans are selected at random, find:

a the expected number of black jelly beans

b the standard deviation

c the variance.

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9 A set of cards contains 5 blue and 7 white cards. If 3 are drawn out at random, the discrete random variable X is the number of blue cards drawn out. Find the mean and variance of X if the cards are drawn out:

a with replacement

b without replacement

10 In a game, 2 cards are drawn from a deck of 52 standard playing cards. A player wins 5 points if one of the cards is an ace and 10 points for double aces.

a If random variable X is the number of aces drawn:

i create a probability distribution for X

ii find the mean, variance and standard deviation for this distribution.

b If random variable Y is the number of points won:

i create a probability distribution for Y

ii find the mean, variance and standard deviation for this distribution.

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10. TEST YOURSELF

For Questions **1** to **4**, select the correct answer **A**, **B**, **C** or **D**.

Qz

Practice quiz

1 The table shows a discrete probability distribution:

x	1	2	3	4	5	6
$P(X = x)$	0.24	0.16	0.08	0.14	0.21	0.17

Find $P(X \geq 2)$.

A 0.4 **B** 0.52 **C** 0.76 **D** 0.48

2 The expected value of the probability distribution below is:

x	1	2	3	4
$p(x)$	0.6	0.1	0.2	0.1

A $0.6 + 0.1 + 0.2 + 0.1$ **B** $\frac{1}{4}(1 \times 0.6 + 2 \times 0.1 + 3 \times 0.2 + 4 \times 0.1)$

C $\frac{1+2+3+4}{4}$ **D** $1 \times 0.6 + 2 \times 0.1 + 3 \times 0.2 + 4 \times 0.1$

3 The value of t in the probability distribution is:

x	1	2	3	4	5	6
$p(x)$	0.28	0.16	0.04	0.1	t	0.25

A 0.13 **B** 0.17 **C** 0.27 **D** 0.07

4 Which table represents a probability function?

A

x	$f(x)$
1	0.6
2	0.2
3	0.1
4	0.2

B

x	$f(x)$
1	0.3
2	0.25
3	0.1
4	0.25

C

x	$f(x)$
1	0.15
2	0.25
3	0.3
4	0.4

D

x	$f(x)$
1	0.25
2	0.4
3	0.15
4	0.2

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5 For each random variable, write the set of possible values.

a The number of 6s when rolling a die 5 times

b The number of heads when tossing a coin 10 times

c The first day the temperature rises above 28° in November

d The number of doubles when rolling 2 dice twice

e The number of red cards selected in 9 trials when pulling a card from a hat that contains 20 red and 20 blue cards.

6 This table shows a discrete probability distribution. Evaluate k.

x	0	1	2	3	4
$P(X = x)$	$5k$	$3k$	$4k - 1$	$2k - 3$	$6k$

7 A probability function is given by $p(x) = \frac{x}{15}$ for $x = 1, 2, 3, 4, 5$. Find its mean, variance and standard deviation.

8 The table represents a probability distribution.

x	4	7	8	9
$P(X = x)$	$\frac{1}{6}$	$\frac{5}{12}$	$\frac{1}{3}$	$\frac{1}{12}$

Find:

a $P(X = 9)$ **b** $P(X < 8)$ **c** $P(X \geq 7)$

d $P(4 \leq X \leq 8)$ **e** $P(7 < X \leq 9)$

9 Draw a discrete probability distribution table for the number of tails when tossing 2 coins.

10 State whether each probability distribution is uniform.

a Number of tails when tossing a coin

b Number of heads when tossing 2 coins

c The number rolled on a die

d Number of 6s when rolling a die

11 State whether each random variable is discrete or continuous.

a The number of heads when tossing 5 coins

b The distances between cars parked in the street

c The number of correct answers in an exam

d The masses of babies

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12 Find the expected value, variance and standard deviation for this probability distribution.

x	0	1	2	3	4
$P(X = x)$	31%	22%	18%	24%	5%

13 A spinner has the numbers 1 to 7 evenly spaced around it.

a Draw a probability distribution table for the spinner.

b Is it a uniform distribution?

c Find the probability of spinning a number:

i greater than 5 **ii** 3 or less **iii** at least 4

d Find the expected value of the spinner.

14 A function is given by:

$$f(x) = \begin{cases} \dfrac{x-1}{10} & \text{for } x = 3 \\ \dfrac{x-4}{5} & \text{for } x = 5 \\ \dfrac{x}{15} & \text{for } x = 9 \end{cases}$$

a Find:

i $f(3)$ **ii** $f(5)$ **iii** $f(9)$

b Show that $f(x)$ is a probability function.

15 a Construct a probability distribution table for the number of tails when tossing 2 coins.

b Is it a uniform distribution?

c Find the probability of tossing:

i one tail **ii** at least one tail.

16 State whether each function is a probability function.

a

x	1	2	3	4	5	6
$f(x)$	0.2	0.07	0.15	0.2	0.3	0.08

b $f(x) = \dfrac{x+1}{6}$ for $x = 0, 1, 2, 3$

c $\left(0, \dfrac{1}{8}\right), \left(1, \dfrac{1}{4}\right), \left(2, \dfrac{1}{2}\right), \left(3, \dfrac{1}{16}\right), \left(4, \dfrac{3}{16}\right)$

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17 In a game, Jonas pays \$1 to toss 3 coins together. He wins \$1.50 for 3 heads and \$2 for 3 tails.

a Find the expected value for this game.

b How much would you expect Jonas to win or lose in the long term?

18 **a** Show that the points (3, 21%), (5, 14%), (6, 47%) and (9, 18%) represent a discrete probability function.

b Find $E(X)$ and $Var(X)$.

19 Each table represents a probability distribution. Evaluate n.

a

x	1	2	3	4	5	6
$p(x)$	$\frac{1}{8}$	n	$\frac{1}{16}$	$\frac{3}{8}$	$\frac{5}{16}$	$\frac{1}{16}$

b

x	0	1	2	3
$P(X = x)$	0.27	0.51	0.14	n

c $P(x) = n(2x - 1)$ for $x = 1, 2, 3$

20 The table represents a probability distribution.

x	2	3	4	5	6
$P(X = x)$	0.2	0.2	0.3	a	b

If $E(X) = 3.8$, evaluate a and b.

21 A game involves tossing 2 coins. Tannika wins \$2 for 2 heads or 2 tails and loses \$1 for a head and a tail.

a Draw up a probability distribution table for random variable Y showing the winning amounts.

b If it costs Tannika \$1 to play, would you expect her to win or lose the game?

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10. CHALLENGE EXERCISE

1 For the probability distribution below, $E(X) = 2.94$ and $Var(X) = 2.2564$. Evaluate a, b and c.

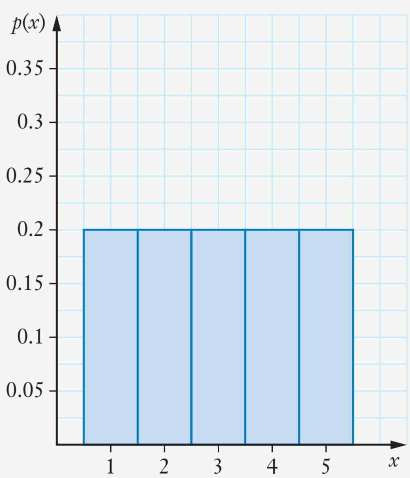
x	1	2	3	4	5
$p(x)$	a	b	c	0.16	0.24

2 The variance of the probability distribution $(1, a)$, $(2, 0.3)$, $(k, 0.4)$, $(5, 0.1)$ is 1.89 and the mean is 2.9. Evaluate a and k.

3 The probability distribution below has $E(X) = 3.34$ and $Var(X) = 4.3044$. Find the value of k and l.

x	1	3	k	6
$P(X = x)$	0.4	0.12	0.3	l

4



a Show that the graph above represents a discrete probability distribution.

b Is it a uniform distribution?

c Find:

i $P(X \leq 3)$ **ii** $P(X > 2)$ **iii** $P(1 \leq X < 5)$

d Find $E(X)$ and $Var(X)$.

e If this distribution changes so that $P(X = 1) = 0.35$, find $P(X = 2)$ if all the other probabilities remain the same.

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5 A sample of people were surveyed to rate a TV show on a scale of 1 to 5.

Rating	Frequency
1	4
2	15
3	23
4	59
5	19

a How many people were surveyed?

b Draw a probability distribution table for the survey results.

c Is the sample mean from the survey a good estimate of the population mean of 2.5?

d Find the standard deviation. Is this a good estimate of the population standard deviation of 1?

e Can you explain these results from **c** and **d**?

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