9 Graphs of practical situations

Syllabus topic — A3.2 Graphs of practical situations

This topic will develop your skills in graphing non-linear functions and how they can be used to model and solve a range of practical problems.

Outcomes

- Construct a graph of an exponential function using a table of ordered pairs.
- Use an exponential model to solve a practical problem.
- Construct a graph of an quadratic function using a table of ordered pairs.
- Use a quadratic model to solve a practical problem.
- Construct a graph of an reciprocal function using a table of ordered pairs.
- Use a reciprocal model to solve a practical problem.

Digital Resources for this chapter

In the Interactive Textbook:

- Videos
- Literacy worksheet
- Quick Quiz
- Solutions (enabled by teacher)
- Desmos widgets
- Spreadsheets
- Study guide

In the Online Teaching Suite:

- Teaching Program
- Tests
- Review Quiz
- Teaching Notes

Knowledge check

The Interactive Textbook provides a test of prior knowledge for this chapter, and may direct you to revision from the previous years' work.

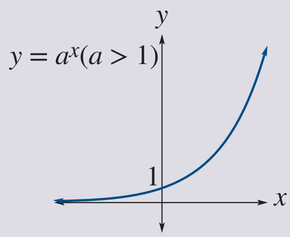
ISBN 978-1-108-44806-2

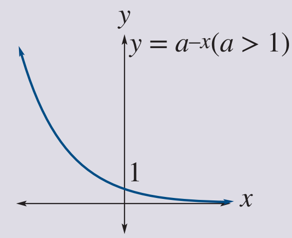
9A Graphs of exponential functions

Exponential functions have x as the power of a constant (e.g. 3^x). They are defined by the general rule $y = a^x$ and $y = a^{-x}$ where the constant $a > 0$. Their graphs are shown below.

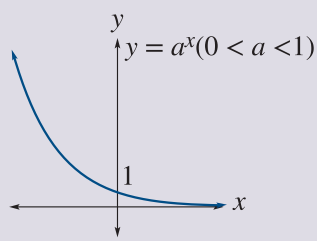
GRAPHS OF EXPONENTIAL FUNCTIONS

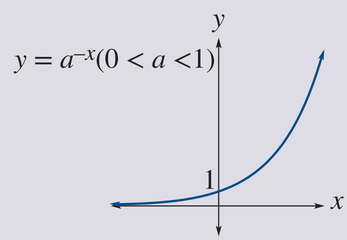
Most practical uses of exponential functions have $a > 1$:





When a is greater than 0 but less than 1, the shape of the curve is reversed horizontally:





When $a = 1$ the graph is flat line, $y = 1$.

Key features of exponential graphs

- The graph lies wholly above the x-axis because a^x is always positive, regardless of the value of x. It is impossible for the y values to be zero or negative.
- The graph always passes through the point (0, 1) because when $x = 0$ then $y = a^0 = 1$, regardless of the value of a.
- The x-axis is an asymptote. That is, it is a line that the curve approaches by getting closer and closer to it but never reaching it.
- The graph $y = a^{-x}$ is the reflection of the graph $y = a^x$ about the y-axis.
- Increasing the value of a such as changing $y = 2^x$ to $y = 3^x$ affects the steepness of the graph. The y values increase at a greater rate when the x values increase.
- The exponential function $y = a^x$ when $a > 1$ is often referred to as a growth function because as the x values increase, the y values increase.
- The exponential function $y = a^{-x}$ when $a > 1$ is often referred to as a decay function because as the x values increase, the y values decrease.

To graph an exponential function:

1. Construct a table of values.
2. Draw a number plane.
3. Plot the points.
4. Join the points to make a curve.

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Example 1: Graphing an exponential function 9A

Draw the graph of $y = 3^x$.

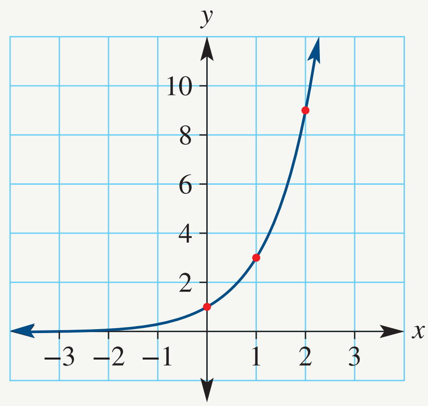
SOLUTION:

1. Construct a table of values for x and y.
2. Let $x = -3, -2, -1, 0, 1, 2$ and 3. Find y using the exponential function $y = 3^x$.

x	-3	-2	-1	0	1	2	3
y	$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27

3. Draw a number plane with x as the horizontal axis and y as the vertical axis.
4. Plot the points $(-3, \frac{1}{27})$, $(-2, \frac{1}{9})$, $(-1, \frac{1}{3})$, $(0, 1)$, $(1, 3)$, $(2, 9)$ and $(3, 27)$.
5. Join the points to make a curve.

Note: Sometimes it is necessary to rescale the axes to plot the points as some points are impractical to plot, such as $(-3, \frac{1}{27})$.



Example 2: Graphing an exponential function 9A

Draw the graph of $y = 3^{-x}$.

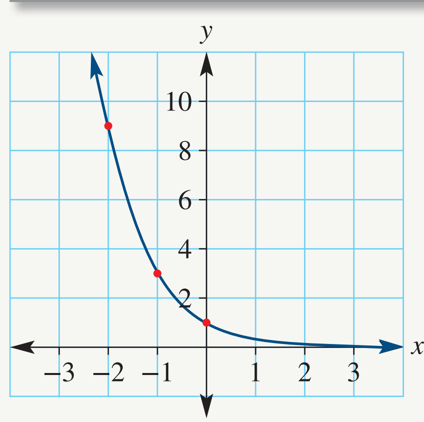
SOLUTION:

1. Construct a table of values for x and y.
2. Let $x = -3, -2, -1, 0, 1, 2$ and 3. Find y using the exponential function $y = 3^{-x}$.

x	-3	-2	-1	0	1	2	3
y	27	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{27}$

3. Draw a number plane with x as the horizontal axis and y as the vertical axis.
4. Plot the points $(-3, 27)$, $(-2, 9)$, $(-1, 3)$, $(0, 1)$, $(1, \frac{1}{3})$, $(2, \frac{1}{9})$ and $(3, \frac{1}{27})$.
5. Join the points to make a curve.

Note: The exponential curve $y = 3^{-x}$ is the reflection of $y = 3^x$ about the y-axis. Both curves pass through $(0, 1)$ and have the x-axis as an asymptote.



Desmos widget 9A Graphing exponential functions with technology

XLS 9A Spreadsheet activity: Graphing exponential functions

ISBN 978-1-108-44806-2

Exercise 9A

Example 1 **1** Complete the following table of values and graph each exponential function.

a $y = 2^x$

x	–2	–1	0	1	2
y		$\frac{1}{2}$			

b $y = 2^{-x}$

x	–2	–1	0	1	2
y					$\frac{1}{4}$

2 Use the graph of $y = 2^x$ in question **1a** to answer these questions.

a Is it possible for y to have negative values?
b Is it possible for x to have negative values?
c Is it possible to calculate y when $x = 0$? If so, what is it?
d Is it possible to calculate x when $y = 0$? If so, what is it?
e What is the approximate value of y when $x = 0.5$?
f What is the approximate value of y when $x = 1.5$?

Example 2 **3** Complete the following table of values by expressing the y values, correct to two decimal places. Graph each exponential function.

a $y = 4^x$

x	–3	–2	–1	0	1	2	3
y	0.02						

b $y = 4^{-x}$

x	–3	–2	–1	0	1	2	3
y						0.06	

4 Use the graph of $y = 4^x$ in question **3a** to answer these questions.

a Is it possible for y to have negative values?
b Is it possible for x to have negative values?
c Is it possible to calculate y when $x = 0$? If so, what is it?
d Is it possible to calculate x when $y = 0$? If so, what is it?
e What is the value of y when $x = 1.5$?
f What is the value of y when $x = 2.5$?

5 Sketch the graph of the following functions on the same set of axes.

a $y = 2^x$ **b** $y = 3^x$ **c** $y = 4^x$

6 Sketch the graph of the following functions on the same set of axes.

a $y = 2^{-x}$ **b** $y = 3^{-x}$ **c** $y = 4^{-x}$

7 What is the effect on the graph of changing the value of a in $y = a^x$?
Hint: Use your graphs in questions **5** and **6**.

ISBN 978-1-108-44806-2

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9B Exponential models

Exponential modelling occurs when a practical situation is described mathematically using an exponential function. The quantity usually experiences fast growth or decay.

EXPONENTIAL MODEL

Exponential growth – Quantity increases rapidly according to the function $y = a^x$ where $a > 1$.

Exponential decay – Quantity decreases rapidly according to the function $y = a^{-x}$ where $a > 1$.

Example 3: Using an exponential model 9B

The fish population is predicted using the formula, $N = 500 \times 1.5^t$ where N is the number of fish and t is the time in years.

a Construct a table of values for t and N. Use values for t from 0 to 4. Approximate the number of fish to the nearest whole number.
b Draw the graph of $N = 500 \times 1.5^t$.
c How many fish were present after 2 years?
d How many extra fish will be present after 4 years compared to 2 years?
e Estimate the number of fish after 18 months.

SOLUTION:

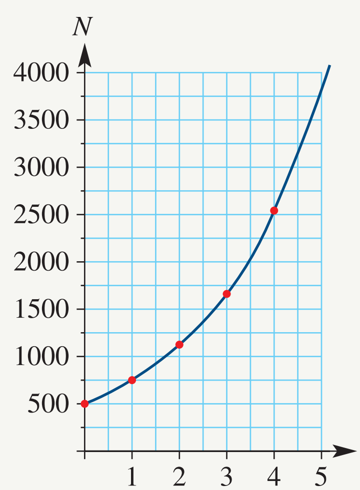
1. Construct a table of values for t and N.
2. Let $t = 0, 1, 2, 3$ and 4. Find N using the exponential function $N = 500 \times 1.5^t$. Express the values for N as a whole number.

a

t	0	1	2	3	4
N	500	750	1125	1688	2531

3. Draw a number plane with t as the horizontal axis and N as the vertical axis.
4. Plot the points (0, 500), (1, 750), (2, 1125), (3, 1688) and (4, 2531).
5. Join the points to make the curve.

b



6. Look up $t = 2$ in the table and find N.

c 1125 fish

7. Subtract the number of fish for 2 years from 4 years.

d Extra = 2531 − 1125 = 1406

8. Read the approximate value of N from the graph when $t = 1.5$.

e Approximately 900 fish

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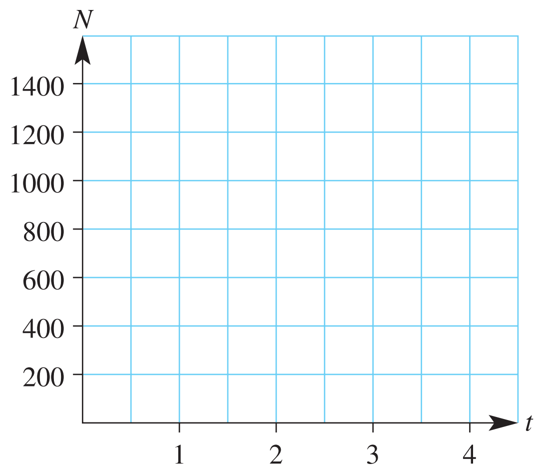
Exercise 9B

Example 3 **1** The exponential function, $N = 6^t$, is used to model the growth in the number of insects (N) after t days.

a Copy and complete the table of values for t and N.

t	0	1	2	3	4
N					

b Copy and draw the graph of $N = 6^t$ on the number plane below.



c What was the initial number of insects?

d How many insects were present after 3 days?

e How many extra insects will be present after 4 days compared with 2 days?

f How many days did it take for the number of insects to exceed 1000?

2 The population of an endangered reptile is decreasing exponentially according to the formula $P = 1.5^{-t} \times 100$, where P is the population of reptiles after t years.

a Copy and complete the table of values for t and P. Express the population of reptiles to the nearest whole number.

t	0	2	4	6	8
N					

b Graph $P = 1.5^{-t} \times 100$ using the table of ordered pairs in part **a**.

c What is the initial population of reptiles?

d Estimate the population of reptiles after 3 years?

e Estimate the population of reptiles after 7 years?

f What is the difference in the population of reptiles after 2 years compared with 6 years?

g Estimate the time taken (to the nearest year) for the population of reptiles to be less than 1.

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3 The size of a flock of birds, F, after t years is decaying exponentially using the function $F = 200 \times 0.5^t$.

a Make a table of values for t and F. Use values for t from 0 to 5. Express F correct to the nearest whole number.

b Draw the graph of $F = 200 \times 0.5^t$.

c What was the initial flock of birds?

d How many birds were present after 6 months (0.5 years)?

e How many birds were present after 3 years?

f How many birds were present after 5 years?

g How many extra birds will be present after 1 year compared with 3 years?

h How many fewer birds will be present after 2 years compared with 4 years?

i How many years will it take for the number of birds to fall to less than one bird?

4 The number of algae grows exponentially according to the function, $b = 30 \times 1.2^t$ where b is the number of algae after t hours.

a Construct a table of ordered pairs using 0, 5, 10, 15 and 20 as values for t. Express the number of algae to the nearest whole number.

b Graph $b = 30 \times 1.2^t$ using the table of ordered pairs in part **a**.

c What is the initial number of algae?

d What is the number of algae after 4 hours?

e What is the number of algae after 8 hours?

f What is the number of algae after 12 hours?

g What is the number of algae after 16 hours?

h Estimate the time taken for the algae to reach 120.

5 Tom invested \$1000 into a managed fund that appreciated in value for 5 years. The amount of money (A) in the fund for each year (t) is shown below.

t	0	1	2	3	4	5
A	\$1000	\$1300	\$1690	\$2197	\$2856	\$3712

a Draw a number plane with t as the horizontal axis and A as the vertical axis.

b Plot the points from the table of values. Join the points to make a curve.

An exponential growth model in the form $y = 1000 \times 1.3^x$ describes this situation.

c Use the model to find the value (to the nearest dollar) of the fund after $2\frac{1}{2}$ years.

d Use the model to find the value (to the nearest dollar) of the fund after 7 years.

e What is the time when the value of the fund is approximately \$1480?

ISBN 978-1-108-44806-2

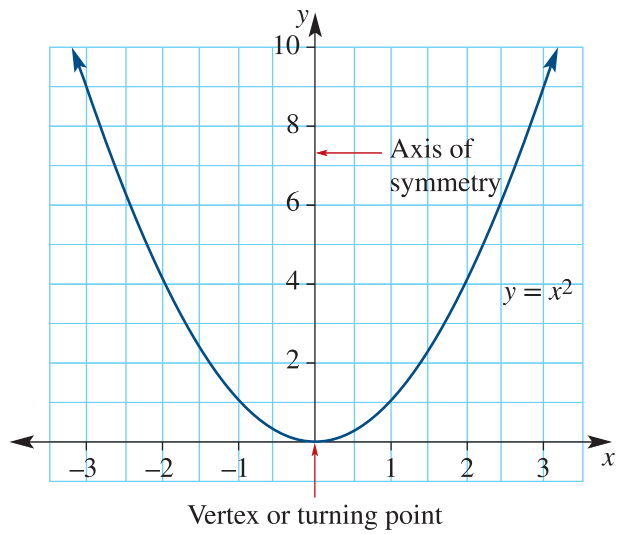
9C Quadratic functions

A quadratic function is a curve whose equation has an x squared (x^2). It is defined by the general rule $y = ax^2 + bx + c$ where a, b and c are numbers. Quadratic functions are graphed in a similar method to exponential functions except the points are joined to make a curve in the shape of a parabola.

Key features of a parabola

The basic parabola has the equation $y = x^2$.

- The vertex (or turning point) is (0, 0).
- It is a minimum turning point.
- Axis of symmetry is $x = 0$ (the y-axis)
- y-intercept is 0 and x-intercept is 0.
- The graph $y = -x^2$ is the reflection of the graph $y = x^2$ about the x-axis.
- Changing the coefficient of the equation such as $y = 2x^2$ or $y = \frac{1}{2}x^2$ affects the height of the parabola.

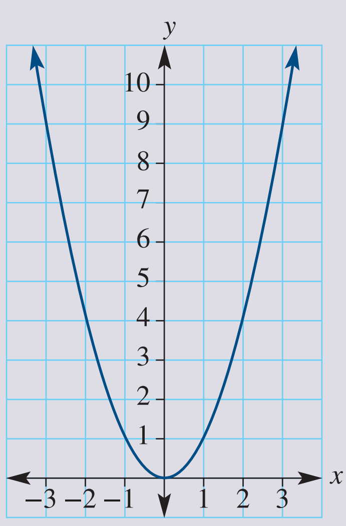


- Adding or subtracting a number to the equation such as $y = x^2 + 1$ or $y = x^2 - 1$ does not change the shape but moves the parabola up or down.

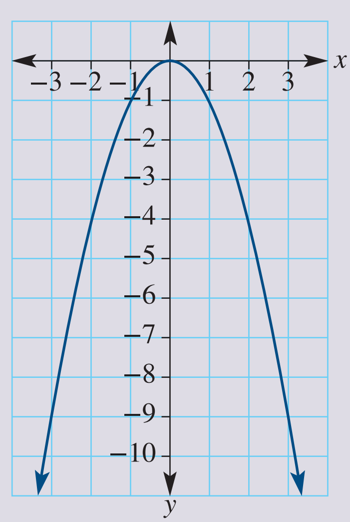
QUADRATIC FUNCTION

A quadratic function has the form $y = ax^2 + bx + c$ where a, b and c are numbers.

Parabola ($y = x^2$) – Minimum turning point



Parabola ($y = -x^2$) – Maximum turning point



To graph a parabola:

1. Construct a table of values.
2. Draw a number plane.
3. Plot the points.
4. Join the points to make a parabola.

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Example 4: Graphing a quadratic function 9C

Draw the graph of $y = x^2 + 1$.

SOLUTION:

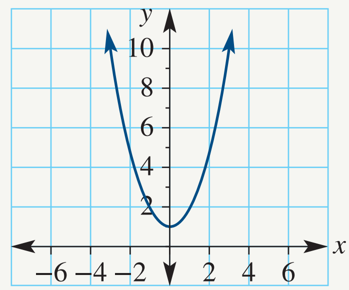
1 Construct a table of values for x and y using $x = -3, -2, -1, 0, 1, 2$ and 3. Find y by substituting into $y = x^2 + 1$.

x	−3	−2	−1	0	1	2	3
y	10	5	2	1	2	5	10

2 Draw a number plane with x as the horizontal axis and y as the vertical axis.

3 Plot the points (−3, 10), (−2, 5), (−1, 2), (0, 1), (1, 2), (2, 5) and (3, 10).

4 Join the points to make a curve in the shape of a parabola.



Example 5: Determining the features of a parabola 9C

Draw the graph of $y = x^2 - 4x + 3$ (use $x = -1, 0, 1, 2, 3, 4, 5$) and find the following features.

a turning point **b** axis of symmetry **c** y-intercept
d x-intercepts **e** minimum value

SOLUTION:

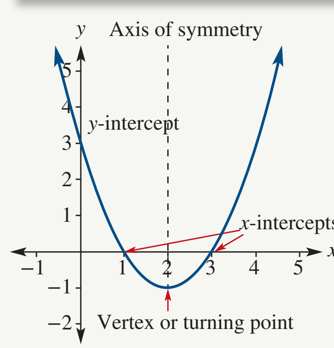
1 Construct a table of values for x and y using $x = -1, 0, 1, 2, 3, 4$ and 5. Find y by substituting into $y = x^2 - 4x + 3$.

x	−1	0	1	2	3	4	5
y	8	3	0	−1	0	3	8

2 Draw a number plane with x as the horizontal axis and y as the vertical axis.

3 Plot the points (−1, 8), (0, 3), (1, 0), (2, −1), (3, 0), (4, 3) and (5, 8).

4 Join the points to make a curve in the shape of a parabola.



5 Find where the graph changes direction. **a** Turning point is (2, −1)

6 Find the line that splits the graph into two. **b** Axis of symmetry is $x = 2$

7 Find the point where the graph cuts the y-axis. **c** y-intercept is 3 (0, 3)

8 Find the point where the graph cuts the x-axis. **d** x-intercepts are 1 and 3

9 Determine the smallest value of y. **e** Minimum value is −1

Desmos widget 9C Graphing a quadratic function with technology

9C Spreadsheet activity: Graphing a quadratic function with a spreadsheet

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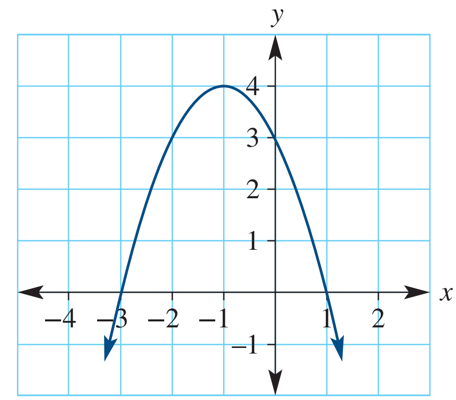
Exercise 9C

1 Write the missing features for this graph.



a The coordinates of the turning point are __________.

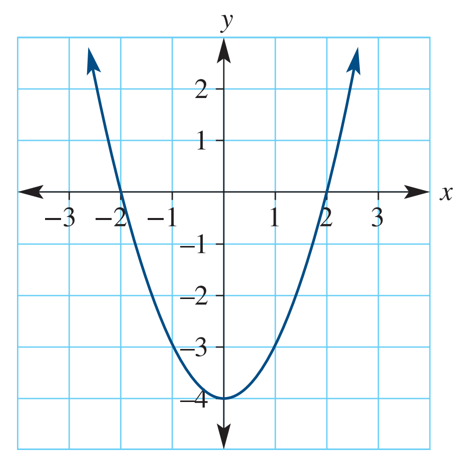
b The y-intercept is __________.

c The x-intercepts are __________ and __________.

d The axis of symmetry is __________.

e The maximum value is __________.

2 Write the missing features for this graph.



a The coordinates of the turning point are __________.

b The y-intercept is __________.

c The x-intercepts are __________ and __________.

d The axis of symmetry is __________.

e The minimum value is __________.

3 Complete the following tables of values and graph each quadratic function.

a $y = x^2$

x	–3	–2	–1	0	1	2	3
y							

b $y = 2x^2$

x	–3	–2	–1	0	1	2	3
y							

c $y = 3x^2$

x	–3	–2	–1	0	1	2	3
y							

d $y = \frac{1}{2}x^2$

x	–3	–2	–1	0	1	2	3
y							

e What is the axis of symmetry for each of the above quadratic functions?

f Is the turning point for each of the above quadratic functions a maximum or minimum?

g What is the effect of changing the coefficient of x^2 in the quadratic equation $y = x^2$?

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4 Complete each table of values and graph the quadratic functions on the same number plane.

a $y = -x^2$

x	–3	–2	–1	0	1	2	3
y							

b $y = -2x^2$

x	–3	–2	–1	0	1	2	3
y							

c $y = -3x^2$

x	–3	–2	–1	0	1	2	3
y							

d $y = -\frac{1}{2}x^2$

x	–3	–2	–1	0	1	2	3
y							

e What is the axis of symmetry for each of the above quadratic functions?

f Is the turning point for each of the above quadratic functions a maximum or minimum?

g What is the effect of changing the coefficient of x^2 in the quadratic equation $y = -x^2$?

Example 4

5 Complete the following table of values and graph each quadratic function on the same number plane.

a $y = x^2 + 1$

x	–3	–2	–1	0	1	2	3
y							

b $y = x^2 - 1$

x	–3	–2	–1	0	1	2	3
y							

c $y = x^2 + 2$

x	–3	–2	–1	0	1	2	3
y							

d $y = x^2 - 2$

x	–3	–2	–1	0	1	2	3
y							

e What is the axis of symmetry for each of the above quadratic functions?

f Is the turning point for each of the above quadratic functions a maximum or minimum?

g What is the effect of adding or subtracting a number to the quadratic function $y = x^2$?

Example 5

6 Complete the following table of values and graph each quadratic function on the same number plane.

a $y = x^2 + 2x + 1$

x	–6	–4	–2	0	2	4	6
y							

b $y = x^2 + 4x + 4$

x	–6	–4	–2	0	2	4	6
y							

c $y = x^2 - 2x + 1$

x	–6	–4	–2	0	2	4	6
y							

d $y = x^2 - 4x + 4$

x	–6	–4	–2	0	2	4	6
y							

e What do all of the above quadratic functions have in common?

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9D Quadratic models

Quadratic modelling occurs when a practical situation is described mathematically using a quadratic function.

QUADRATIC MODEL

A quadratic model describes a practical situation using a function in the form $y = ax^2 + bx + c$ where a, b and c are numbers. Quadratic functions are graphed to make a curve in the shape of a parabola.

Example 6: Using a quadratic model 9D

The area (A) of a rectangular garden of length x metres is given by $A = 6x - x^2$.

x	0	1	2	3	4	5	6
A							

a Draw the graph of $A = 6x - x^2$ using the table of ordered pairs.
b Use the graph to estimate the area of the garden when the length of the garden is 4.5 m.
c What is the maximum area of the garden?
d What is the garden length in order to have maximum area?

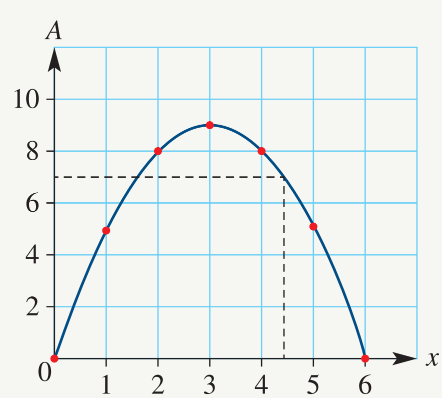
SOLUTION:

1 Let $x = 0, 1, 2, 3, 4, 5$ and 6 and find A by substituting into the quadratic function $A = 6x - x^2$.

a

x	0	1	2	3	4	5	6
A	0	5	8	9	8	5	0

2 Draw a number plane with x as the horizontal axis and A as the vertical axis.
3 Plot the points (0, 0), (1, 5), (2, 8), (3, 9), (4, 8), (5, 5) and (6, 0).
4 Join the points to make a parabolic curve.
5 Draw a vertical line from $x = 4.5$ on the horizontal axis until it intersects the parabola. At this point draw a horizontal line until it connects with the vertical axis.



6 Read this value.

b About 7 m^2. Check your solution algebraically.

$$A = 6x - x^2$$
$$= 6 \times 4.5 - 4.5^2$$
$$= 6.75\,\text{m}^2$$

7 Read the largest value for A.
8 Read the value on the x-axis when the A is largest.

c Maximum area of the garden is 9.
d Maximum area occurs when $x = 3$.

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Exercise 9D

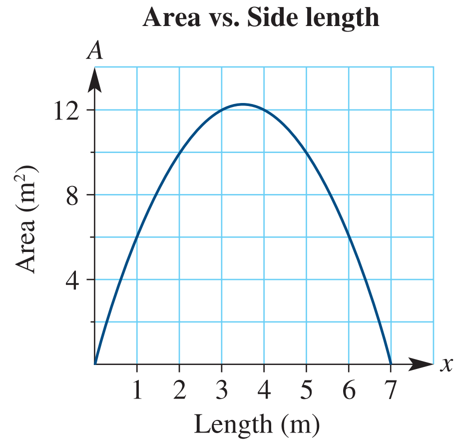
Example 6 **1** The area (A) of a rectangular enclosure of length x metres is given by the formula $A = x(7 - x)$. The graph of this formula is shown opposite.

a What is the area of the enclosure when x is 1 metre?

b What is the area of the enclosure when x is 5 metres?

c What is the enclosure's length in order to have maximum area?

d What is the maximum area of the enclosure?



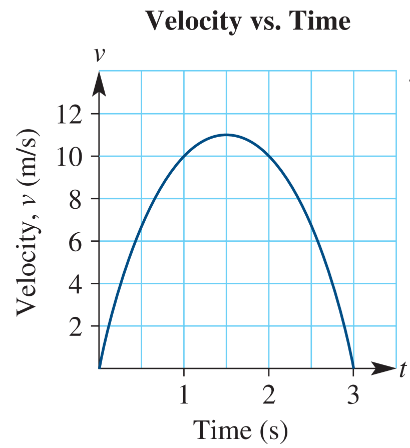
2 The movement of an object with a velocity v (in m/s) at time t (s) is given by the formula $v = 15t - 5t^2$ The graph of this formula is shown opposite.

a What was the initial velocity of the object?

b What was the greatest velocity reached by the object?

c How many seconds did it take for the object to reach maximum velocity?

d Determine the number of seconds when the velocity is greater than 6 m/s. Answer to the nearest second.



3 The price ($\$P$) of rope depends on the diameter (d), in metres, of the rope when it is rolled into a circle. The quadratic equation $P = 3d^2$ is used to model this situation.

a Complete the following table of values, correct to the nearest whole number.

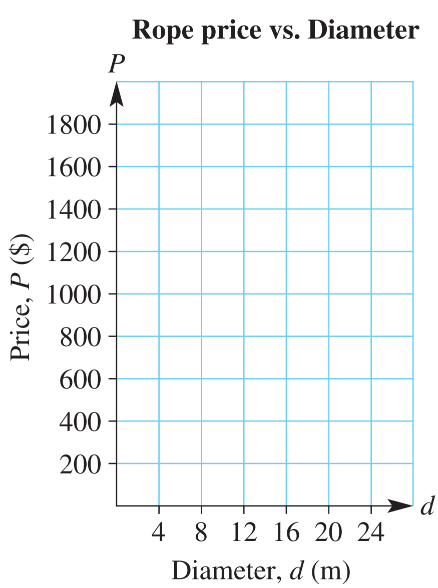
d	0	2	4	6	8	10	12	14	16	18	20	22	24
P													

b Draw the graph of $P = 3d^2$ using the number plane shown opposite.

c What is the price of the rope when the diameter of the rope is 12 metres?

d What is the price of the rope when the diameter of the rope is 23 metres?

e What is the difference between the price of the rope when the diameter of the rope is 5 metres compared with a diameter of 25 metres?



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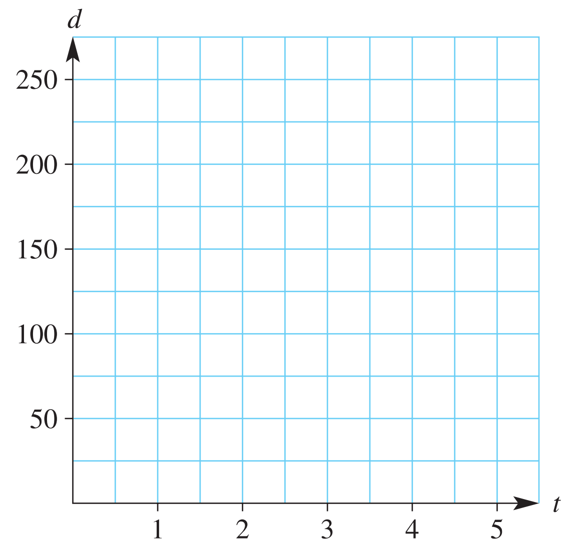
4 A stone falls from rest down a mine shaft. The distance it falls, d metres, at time t seconds is given by the quadratic equation $d = 9.8t^2$.

a Complete the following table of values, correct to the nearest whole number.

t	0	1	2	3	4	5
d						

b Draw the graph of $d = 9.8t^2$ using the number plane shown opposite.



c What is the distance travelled by the stone after 1 second?

d What is the distance travelled by the stone after 5 seconds?

e What is the distance travelled by the stone after 2.5 seconds?

f How long did it take for the stone to travel 100 metres?

g How long did it take for the stone to travel 200 metres?

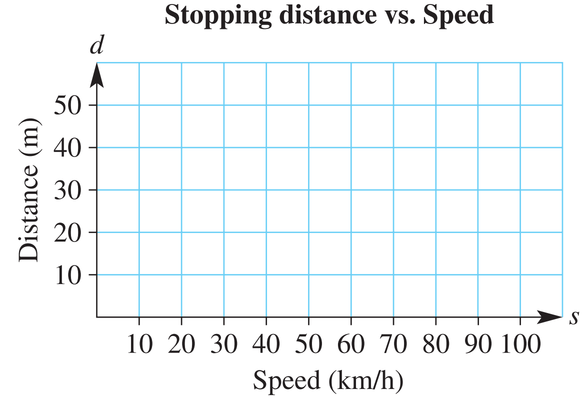
5 The equation $d = 0.005s(s - 1)$ is used to model the stopping distance for a train where d is the stopping distance in metres and s is the train's speed in km/h.

a Complete the following table of values, correct to the nearest whole number.

s	0	20	40	50	60	80	100
d							

b Draw the graph of $d = 0.005s(s - 1)$ using the number plane shown opposite.

Stopping distance vs. Speed



c What is the stopping distance when the train is travelling at 20 km/h?

d What is the stopping distance when the train is travelling at 75 km/h?

e What is the maximum speed (in km/h) a train could be travelling to stop within 15 m?

f What is the maximum speed (in km/h) a train could be travelling to stop within 30 m?

g What is the difference between the stopping distances when a train is travelling at a speed of 40 km/h compared with travelling at a speed of 80 km/h?

h What is the difference between the stopping distances when a train is travelling at a speed of 15 km/h compared with travelling at a speed of 95 km/h?

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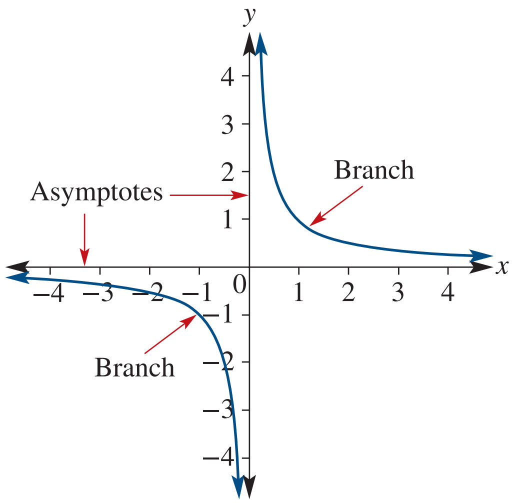
9E Graphs of reciprocal function

A reciprocal function is a curve whose equation has a variable in the denominator such as $\frac{1}{x}$. It is defined by the general rule $y = \frac{k}{x}$ where k is a number. Reciprocal functions are graphed in a similar method to other non-linear functions and make a curve called a hyperbola.

Key features of a hyperbola

The basic hyperbola has the equation $y = \frac{1}{x}$.

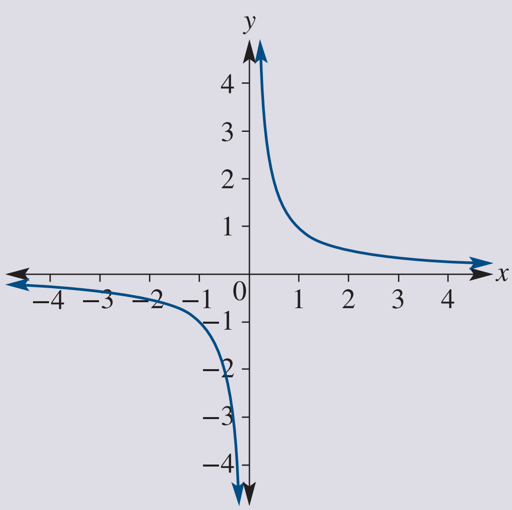
- No value exists for y when $x = 0$.
- The curve has two parts called branches. Each branch is the same shape and size; they are symmetrical and are in opposite quadrants.
- The x-axis and the y-axis are asymptotes of the curve. That is, the curve approaches the x-axis and the y-axis but never touches them.
- The asymptotes are at right angles to each other, so the curve is also called a rectangular hyperbola.



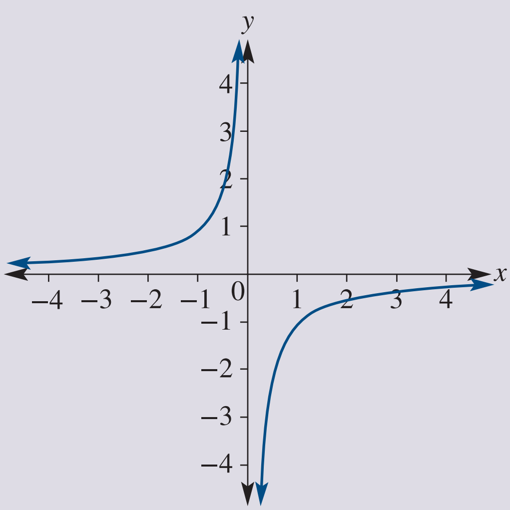
RECIPROCAL FUNCTION

A reciprocal function has the form $y = \frac{k}{x}$ where k is a number.

Hyperbola: $y = \frac{1}{x}$



Hyperbola: $y = -\frac{1}{x}$



To graph a hyperbola:

1. Construct a table of values.
2. Draw a number plane.
3. Plot the points.
4. Join the points to make a hyperbola.

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Example 7: Graphing a reciprocal function 9E

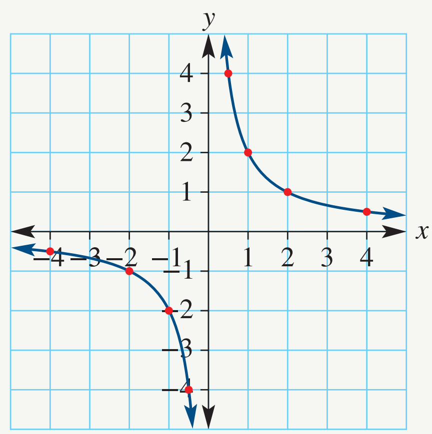
Draw the graph of $y = \frac{2}{x}$.

SOLUTION:

1 Construct a table of values for x and y.
2 Let $x = -4, -2, -1, -0.5, 0.5, 1, 2$ and 4. Find y using the reciprocal function.

x	-4	-2	-1	-0.5	0.5	1	2	4
y	-0.5	-1	-2	-4	4	2	1	0.5

3 Draw a number plane with x as the horizontal axis and y as the vertical axis.
4 Plot the points $(-4, -0.5)$, $(-2, -1)$, $(-1, -2)$, $(-0.5, -4)$, $(0.5, 4)$, $(1, 2)$, $(2, 1)$ and $(4, 0.5)$.
5 No value exists for y when $x = 0$. This results in the curve having two branches.
6 Join the points to make a curve in the shape of a hyperbola.



Example 8: Graphing a reciprocal function 9E

a Draw the graph of $y = -\frac{2}{x}$.

b What are the asymptotes for this graph?

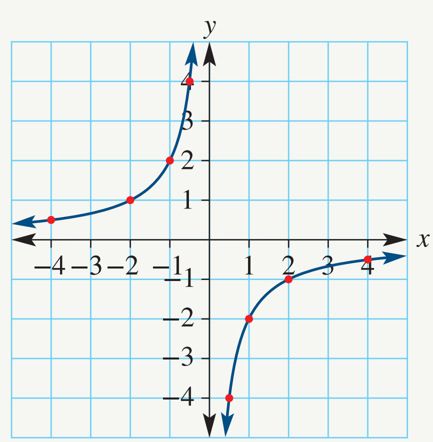
SOLUTION:

1 Construct a table of values for x and y.
2 Let $x = -4, -2, -1, -0.5, 0.5, 1, 2$ and 4. Find y using the reciprocal function.

a

x	-4	-2	-1	-0.5	0.5	1	2	4
y	0.5	1	2	4	-4	-2	-1	-0.5

3 Draw a number plane with x as the horizontal axis and y as the vertical axis.
4 Plot the points $(-4, 0.5)$, $(-2, 1)$, $(-1, 2)$, $(-0.5, 4)$, $(0.5, -4)$, $(1, -2)$, $(2, -1)$ and $(4, 0.5)$.
5 No value exists for y when $x = 0$. This results in the curve having two branches.
6 Join the points to make a curve in shape of a hyperbola.



7 The curve approaches the x-axis and the y-axis but never touches them.

b Asymptotes are $x = 0$ and $y = 0$.

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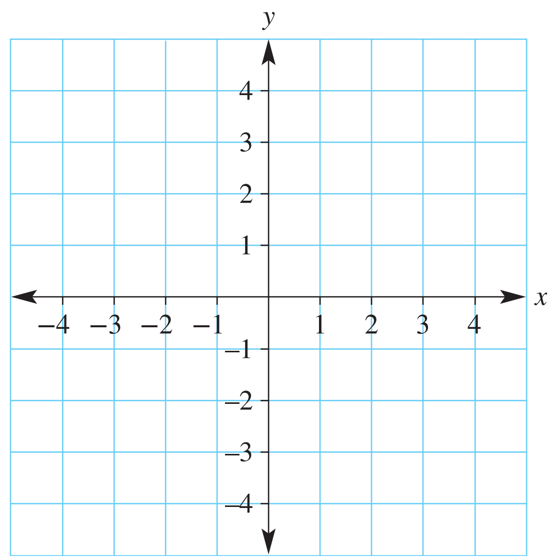
Exercise 9E

Example 7 **1** A reciprocal function is $y = \frac{1}{x}$.

a Complete the following table of values.

x	–4	–2	–1	$-\frac{1}{2}$	$\frac{1}{2}$	1	2	4
y								

b Graph the reciprocal function using the number plane opposite.

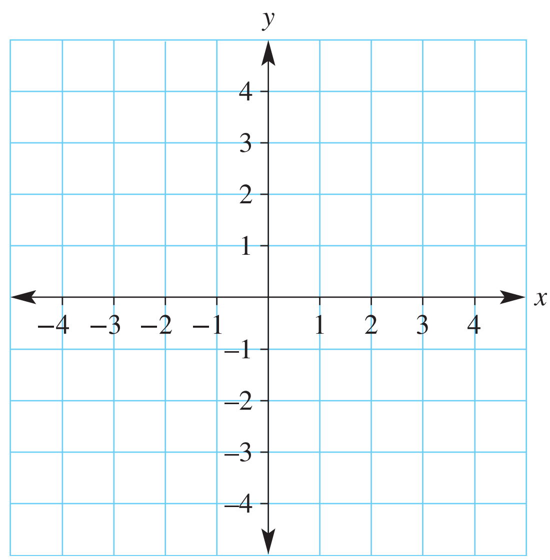


Example 8 **2** A reciprocal function is $y = -\frac{1}{x}$.

a Complete the following table of values.

x	–4	–2	–1	$-\frac{1}{2}$	$\frac{1}{2}$	1	2	4
y								

b Graph the reciprocal function using the number plane opposite.



3 Complete the following table of values and graph each reciprocal function on the same number plane.

a $y = \frac{3}{x}$

x	–9	–3	–1	$-\frac{1}{3}$	$\frac{1}{3}$	1	3	9
y								

b $y = -\frac{3}{x}$

x	–9	–3	–1	$-\frac{1}{3}$	$\frac{1}{3}$	1	3	9
y								

c $y = \frac{4}{x}$

x	–8	–4	–1	$-\frac{1}{2}$	$\frac{1}{2}$	1	4	8
y								

d $y = -\frac{4}{x}$

x	–8	–4	–1	$-\frac{1}{2}$	$\frac{1}{2}$	1	4	8
y								

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9F Reciprocal models

Reciprocal modelling occurs when a practical situation is described mathematically using a reciprocal function. The quantity usually experiences fast growth or decay.

RECIPROCAL MODELS

A reciprocal model describes a practical situation using a function in the form $y = \frac{k}{x}$ where k is a number. Reciprocal functions are graphed to make a curve in the shape of a hyperbola.

Example 9: Using a reciprocal model 9F

The time taken (t), in hours, for a road trip, at speed (s), in km/h, is given by the reciprocal function $t = \frac{2000}{s}$.

a Construct a table of values for s and t.

b Draw the graph of $t = \frac{2000}{s}$.

c How long did the road trip take at a speed of 70 km/h?

d Why is it impossible to complete the road trip in 10 hours?

SOLUTION:

1 Construct a table of values for s and t.

2 Choose appropriate values for s, the speed of the car. Let $s = 10, 20, 40, 50, 60, 80$ and 100.

3 Find t using the reciprocal function $t = \frac{2000}{s}$. Express the values for t as a whole number.

a

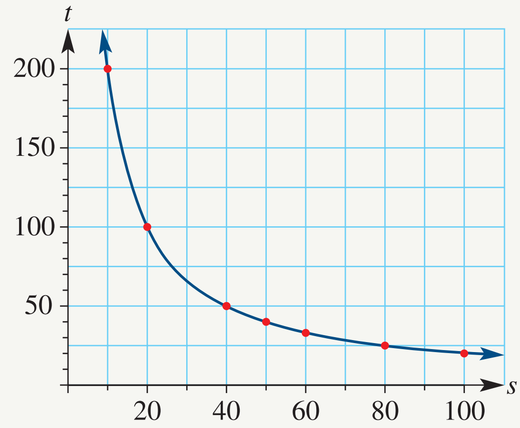
s	10	20	40	50	60	80	100
t	200	100	50	40	33	25	20

4 Draw a number plane with s as the horizontal axis and t as the vertical axis.

5 Plot the points (10, 200), (20, 100), (40, 50), (50, 40), (60, 33), (80, 25) and (100, 20).

6 Join the points to make a branch of a hyperbola.

b



7 Read the approximate value of t from the graph when $s = 70$.

c Approximately 30 hours

8 Read the value of s from the table.

9 Make sense of the result.

d Speed required to complete the trip in 10 h is 200 km/h, which is above the speed limit on Australian roads.

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Example 10: Using a reciprocal model 9F

The cost per person of sharing a pizza ($\$C$) is dependent on the number of people (n) eating the pizza.

The reciprocal equation $C = \frac{24}{n}$ is used to model this situation.

a Describe the possible values for n.
b Construct a table of values for n and C.
c Draw the graph of $C = \frac{24}{n}$.
d What is the cost per person if six people are sharing a pizza?
e How many people shared a pizza if the cost was $2.40 per person?

SOLUTION:

1 The variable n represents the number of people sharing a pizza.

a n is a positive whole number and likely to be less than 10.

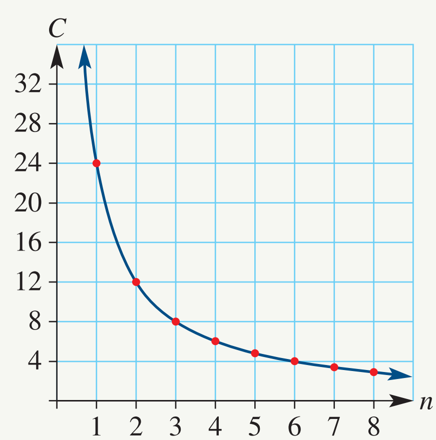
2 Construct a table of values for n and C.
3 Choose appropriate values for n.
Let $n = 1, 2, 3, 4, 5, 6, 7$ and 8.
4 Find C using $C = \frac{24}{n}$.

b

n	1	2	3	4	5	6	7	8
C	24	12	8	6	4.8	4	3.4	3

5 Draw a number plane with n as the horizontal axis and C as the vertical axis.
6 Plot the points $(1, 24)$, $(2, 12)$, $(3, 8)$, $(4, 6)$, $(5, 4.8)$, $(6, 4)$, $(7, 3.4)$ and $(8, 3)$.
7 Join the points to make a branch of a hyperbola.

c



8 Read the value of C from the table or graph when $n = 6$.

d Cost per person is $4.

9 Substitute 2.4 for C into the reciprocal equation.
10 Solve the equation for n by rearranging the formula and evaluate.
11 Check that the answer is reasonable.
12 Write the answer in words.

e $2.4 = \frac{24}{n}$

$n = \frac{24}{2.4} = 10$

$\therefore$ number of people sharing the pizza was 10

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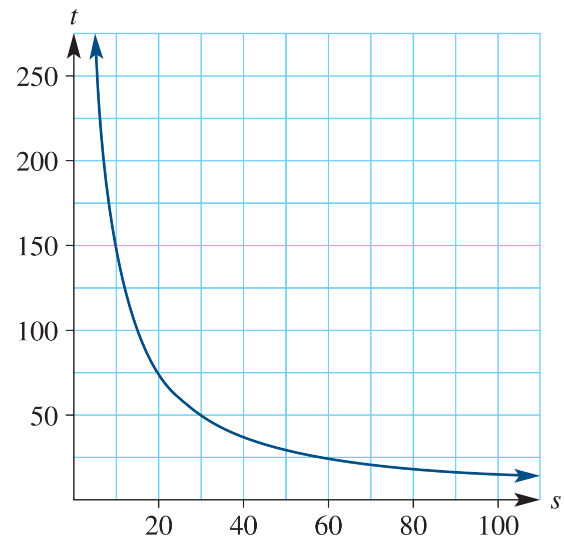
Exercise 9F

Example 9 **1** The time taken (t), in hours, for a road trip, at speed (s), in km/h, is given by the formula $t = \frac{1500}{s}$. The graph of this formula is shown opposite.



a How long did the road trip take at a speed of 50 km/h?

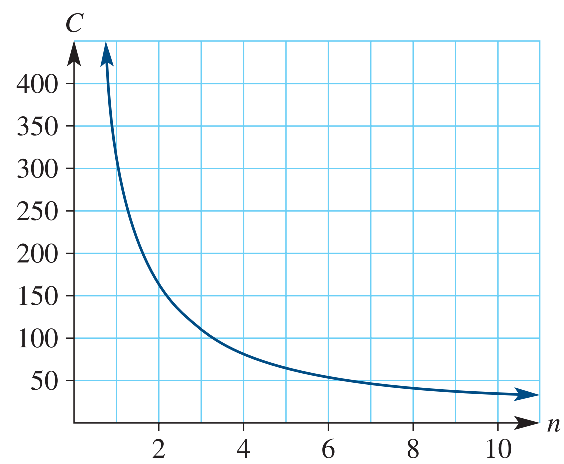
b How long did the road trip take at a speed of 75 km/h?

c What is the speed required to complete the road trip in 25 hours?

d What is the speed required to complete the road trip in 100 hours?

e Why is it impossible to complete the road trip in 5 hours?

Example 10 **2** The cost per person of hiring a yacht (\$$C$) is dependent on the number of people (n) sharing the total cost. The reciprocal equation $C = \frac{320}{n}$ is used to model this situation.



a What is the cost per person of hiring the yacht if 2 people share the total cost?

b What is the cost per person of hiring the yacht if 8 people share the total cost?

c How many people are required to share the cost of hiring a yacht for \$80?

d How many people are required to share the cost of hiring a yacht for \$320?

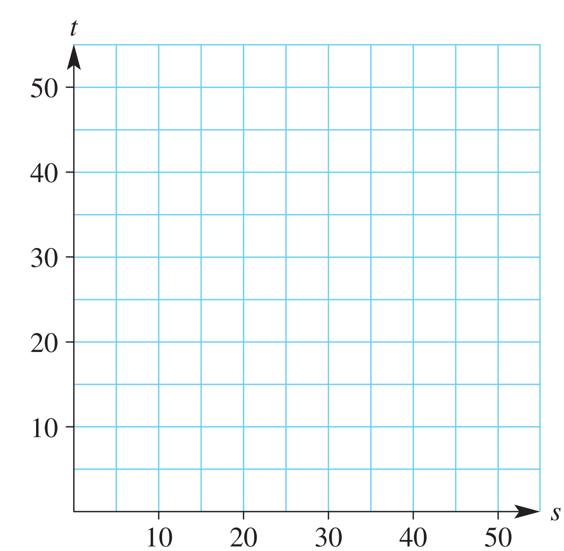
e Is it possible for the cost per person to be \$1?

3 The time taken (t in minutes) to type an essay depends on the typing speed (s in words per minute). The reciprocal function $t = \frac{150}{s}$ is used to model this situation.

a Complete the following table of values, correct to the nearest whole number.

s	5	10	15	25	30	50
t						

b Draw the graph of $t = \frac{150}{s}$ using the number plane shown opposite.



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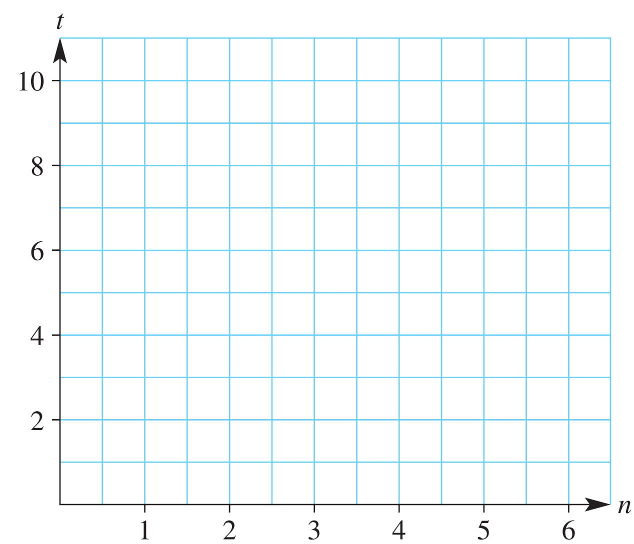
4 The time taken (t in hours) to dig a hole is dependent on the number of people (n) digging the hole. This relationship is modelled using the formula $t = \frac{6}{n}$.

a Complete the following table of values, correct to one decimal place.

n	1	2	3	4	5	6
t						

b Draw the graph of $t = \frac{6}{n}$ using the number plane shown opposite.



c What is the time taken to dig a hole by 1 people?

d What is the time taken to dig a hole by 3 people?

e What is the time taken to dig a hole by 6 people?

f How many people could dig the hole in two hours?

g How many people could dig the hole in 30 minutes?

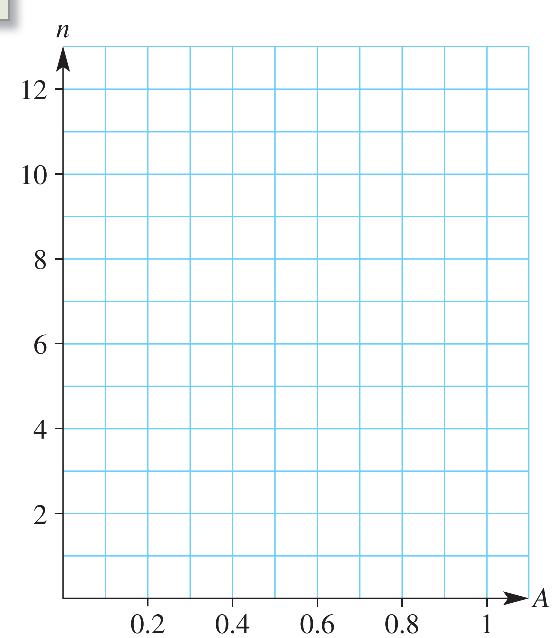
h How long would it take for 360 people to dig the hole? Is this possible?

5 The maximum number of people (n in 1000s) attending an outdoor concert is dependent on the area (A in m^2) allowed per person. The reciprocal equation $n = \frac{1.2}{A}$ models this practical situation.

a Complete the following table of values, correct to the nearest whole number.

A	0.1	0.2	0.3	0.4	0.5	0.6	0.8	0.9	1.0
n									

b Draw the graph of $n = \frac{1.2}{A}$ using the number plane shown opposite.



c How many people can attend this concert if the area allowed is $0.5\,m^2$?

d How many people can attend this concert if the area allowed is $0.25\,m^2$?

e What is the area allowed per person if the maximum number of people attending the concert is 2000?

f What is the area allowed per person if the maximum number of people attending the concert is 5000?

g Is it possible for 12 000 people to attend this concert? Justify your answer.

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9G Miscellaneous problems

Algebraic modelling occurs when a practical situation is described mathematically using an algebraic function. This involves gathering data and analysing the data to determine possible functions. Determining the function is made easier using technology.

ALGEBRAIC MODEL

- Algebraic models are used to describe practical situations.
- Algebraic models may have limitations that restrict their use.

Example 11: Modelling physical phenomena 9G

The mass M kg of a baby orang-utan and its age after x months are given below.

x	0	1	2	3	4	5	6
M	1.5	1.8	2.2	2.6	3.1	3.7	4.5

a Plot the points from the table onto a number plane.

b The formula $M = 1.5(1.2)^x$ models the data in the table. Graph $M = 1.5(1.2)^x$ on the same number plane.

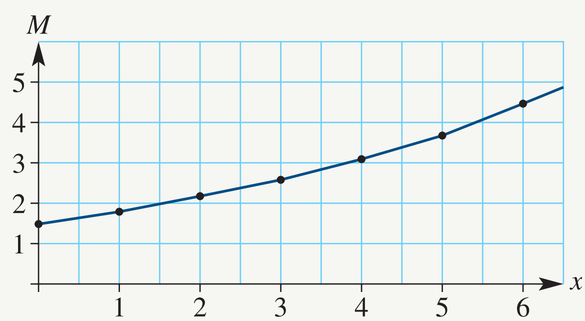
c Use the model to determine the mass of the orang-utan after 2.5 months.

d This model only applies when x is less than or equal to 6. Why?

SOLUTION:

1. Draw a number plane with x as the horizontal axis and M as the vertical axis.
2. Plot the points (0, 1.5), (1, 1.8), (2, 2.2), (3, 2.6), (4, 3.1), (5, 3.7) and (6, 4.5).
3. The formula $M = 1.5(1.2)^x$ has the same table of values. Join the points to make a curve.

a, b



4. Substitute 2.5 for x into the formula.
5. Evaluate, correct to one decimal place.

c $M = 1.5(1.2)^{2.5}$

$= 2.4$ kg

6. Use the model for x greater than 6. Let x be 48 months or 4 years. Substitute 48 for x into the formula.
7. Evaluate.
8. Write the answer in words.

d $M = 1.5(1.2)^{48}$

$= 9479.6$ kg

Orang-utans are less than 100 kg in general so the answer of 9479.6 kg is unreasonable.

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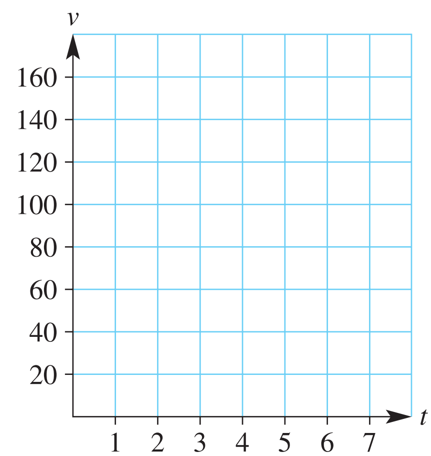
Exercise 9G

1 A new piece of equipment is purchased by a business for \$150000. The value of the equipment (v in \$1000), to the nearest whole number, is depreciated each year (t) using the table below.

t	0	1	2	3	4	5	6
v	150	75	38	19	9	5	2



a Draw a number plane shown opposite.

b Plot the points from the table of values. Join the points to make a curve.

An exponential model in the form $v = 2^{-t} \times 150$ describes this situation.

c Use the model to predict the value of the equipment after 1.5 years.

d Use the model to predict the value of the equipment after 2.5 years.

e Use the model to predict the value of the equipment after 3.5 years.

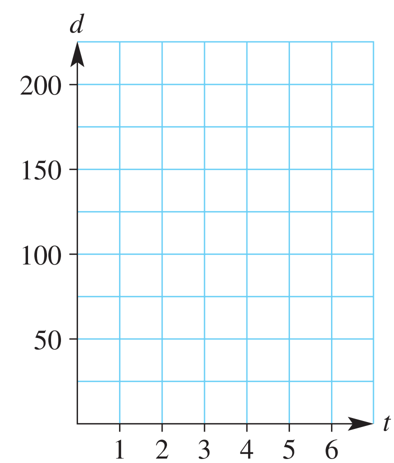
f Use the model to predict the value of the equipment after 6 months.

g When will the value of the equipment be \$75000?

h Use the model to predict the value of the equipment after 20 years. Explain your answer.

2 The distance (d metres) that an object falls in t seconds is shown in the table below.

t	0	1	2	3	4	5	6
d	0	5	20	45	80	125	180



a Plot the points from the table on the number plane. Join the points to make a curve.

A quadratic model in the form $d = 5t^2$ describes this situation.

b Use the model to find the distance fallen after 1.5 seconds.

c Use the model to find the distance fallen after 2.5 seconds.

d Use the model to find the distance fallen after 3.5 seconds.

e Use the model to find the distance fallen after 10 seconds.

f What is the time taken for an object to fall 320 metres?

g Earth's atmosphere is approximately 100 km. What limitation would you place on this model?

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3 The number of tadpoles (N) in a pond after t months is shown in the table below.

t	0	2	4	6	8	10	12	14
N	0	24	96	216	384	600	864	1176

a Draw a number plane with t as the horizontal axis and N as the vertical axis.

b Plot the points from the table of values. Join the points to make a curve.

A quadratic model in the form $N = 6t^2$ describes this situation.

c Use the model to find the number of tadpoles after 3 months.

d Use the model to find the number of tadpoles after 5 months.

e Use the model to find the number of tadpoles after 7 months.

f Use the model to find the number of tadpoles after 11 months.

g Use the model to find the time taken for the number of tadpoles to reach 2400.

h Use the model to predict the number of tadpoles after 4.5 months. What limitations would you place on this model?

4 The speed of a car (s in km/h) and the time taken (t in hours) is shown below.

t	1	2	3	4	5	6
s	120	60	40	30	24	20

a Draw a number plane with t as the horizontal axis and s as the vertical axis.

b Plot the points from the table of values. Join the points to make a curve.

A hyperbolic model in the form $s = \dfrac{120}{t}$ describes this situation.

c Use the model to find the speed of the car if time taken is 1.5 seconds.

d Use the model to find the speed of the car if time taken is 2.5 seconds.

e Use the model to find the speed of the car if time taken is 3.5 seconds.

f Use the model to find the speed of the car if time taken is 8 seconds.

g What is the time taken if the car is travelling at a speed of 48 km/h?

h Use the model to predict the speed of the car after $\frac{1}{2}$ second. Is this possible? Explain your answer.

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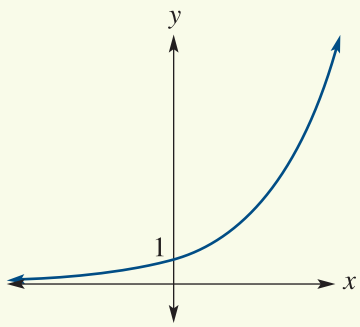
PPT
STUDY GUIDE

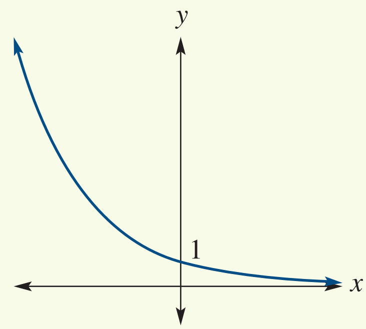
Key ideas and chapter summary

Exponential function

$y = a^x,\ a > 1$



$y = a^{-x},\ a > 1$



Exponential model

Exponential growth

Quantity increases rapidly using $y = a^x$

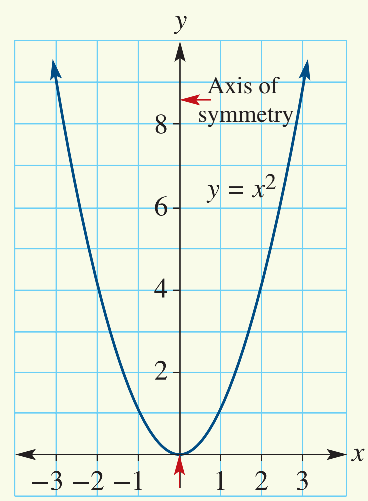
Exponential decay

Quantity decreases rapidly using $y = a^{-x}$

Quadratic function

Quadratic function has the form $y = ax^2 + bx + c$ where a, b and c are numbers.

- Parabola ($y = x^2$)
  Minimum turning point
- Parabola ($y = -x^2$)
  Maximum turning point



Vertex or turning point

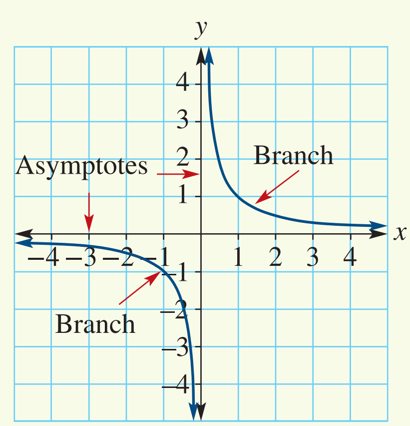
Quadratic model

A quadratic model describes a practical situation using a function in the form $y = ax^2 + bx + c$, where a, b and c are numbers.

Reciprocal function

A reciprocal function has the form $y = \dfrac{k}{x}$, where k is a number.

- Hyperbola: $y = \dfrac{1}{x}$
- Hyperbola: $y = -\dfrac{1}{x}$



Reciprocal model

A reciprocal model describes a practical situation using a function in the form, $y = \dfrac{k}{x}$ where k is a number.

ISBN 978-1-108-44806-2

Review

Multiple-choice

1 What is the y-intercept of the exponential function $y = 2^{-x}$?

A (0, −1) **B** (0, 0) **C** (0, 1) **D** (0, 2)

2 Which of the following points lies on the quadratic curve $y = 2x^2$?

A (−1, 0) **B** (0, −1) **C** (1, 2) **D** (2, 16)

3 What is the maximum y value of the quadratic function $y = -x^2 + 4x - 3$?

x	−1	0	1	2	3	4	5
y							

A −3 **B** 1 **C** 2 **D** 4

4 The equation $d = 0.4(s^2 + s)$ is used to model the stopping distance for a bicycle where d is the stopping distance in metres and s is the bicycle's speed in m/s. What is the stopping distance given a speed of 5 metres per second?

A 5 m **B** 10 m **C** 12 m **D** 15 m

5 Which of the following points lies on the reciprocal function $y = \frac{8}{x}$?

A (−2, 8) **B** (−1, 8) **C** (0, 8) **D** (2, 4)

6 The speed in km/h (s) of a vehicle is given by the formula $s = \frac{200}{t}$ where t is the time in hours. What is the time taken if the average speed was 100 km/h?

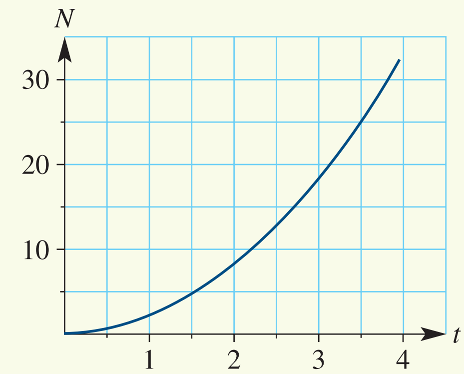
A 0.4 hours **B** 2 hours **C** 100 hours **D** 300 hours

7 A hyperbola has the equation $y = \frac{2}{x}$. Which of the following is an equation of the asymptote?

A $x = 0$ **B** $x = 1$ **C** $x = 2$ **D** $x = \frac{2}{y}$

8 The graph opposite shows the insect population (N) plotted against the time (t) in days.

t	0	1	2	3	4
N	0	2	8	18	32



What type of function would model this data?

A Exponential

B Hyperbolic

C Quadratic

D Reciprocal

ISBN 978-1-108-44806-2

Cambridge University Press

Review

Short-answer

1 Complete the following table of values by expressing the y values, correct to one decimal place. Graph each exponential function.

a $y = 1.5^x$

x	–3	–2	–1	0	1	2	3
y							

b $y = 0.5^x$

x	–3	–2	–1	0	1	2	3
y							

2 The height h cm of a plant and its age after x months is given below.

x	0	1	2	3	4	5	6
h	1.1	2.4	5.3	11.7	25.8	56.7	124.7

a Plot the points from the above table onto a number plane.

b The formula $h = 2.2^x \times 1.1$ models the data in the table. Draw this function.

c Use the model to determine the height of a plant after 1.5 months. Answer correct to one decimal place.

d Use the model to determine the height of a plant after 3.5 months. Answer correct to one decimal place.

3 The population of earthworms grows exponentially according to the formula $w = 1.1^t \times 25$, where w is the number of earthworms after t days.

a Construct a table of ordered pairs using 0, 5, 10, 15 and 20 as values for t. Express the number of earthworms to the nearest whole number.

b Graph $w = 1.1^t \times 25$ using the table of ordered pairs in part **a**.

c What is the initial number of earthworms?

d What is the number of earthworms after 3 days?

e Estimate the time taken for the earthworms to reach a population of 75.

4 Complete the following table of values and graph each quadratic function.

a $y = 3x^2$

x	–3	–2	–1	0	1	2	3
y							

b $y = -\frac{1}{3}x^2$

x	–9	3	–1	0	1	3	9
y							

c $y = x^2 + 3$

x	–3	–2	–1	0	1	2	3
y							

d $y = x^2 - 5x - 4$

x	0	1	2	3	4	5	6
y							

ISBN 978-1-108-44806-2

Cambridge University Press

5 Abbey throws a rock and it takes 6 seconds to reach the ground. The height it reaches is given by the formula $h = -t^2 + 6t$ a where h is the height (in metres) and t is the number of seconds after it has been thrown.

a Complete the following table of values.

t	0	1	2	3	4	5	6
h							

b Draw the graph of $h = -t^2 + 6t$.

c What was the maximum height reached by the rock?

d When was the maximum height reached?

6 Complete the following table of values and graph each reciprocal function on the same number plane.

a $y = \frac{7}{x}$

x	-7	-1	$-\frac{1}{7}$	$\frac{1}{7}$	1	7
y						

b $y = -\frac{7}{x}$

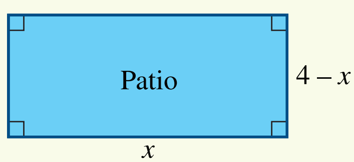
x	-7	-1	$-\frac{1}{7}$	$\frac{1}{7}$	1	7
y						

7 The number of chairs (c) in a row varies inversely with the distance (d in metres) between them. When the chairs are 2 m apart the row can accommodate 60 chairs.

a How many chairs can be placed in a row if the distance between them is 1.5 m?

b What is the distance between the chairs if the number of chairs is 40?

8 A rectangular patio has a length of x metres and a breadth of $(4 - x)$ metres.



a Show that the area of the patio is $A = x \times (4 - x)$.

b Complete the table using the above equation.

x	0	0.5	1	1.5	2	2.5	3	3.5	4
A									

c Draw the graph of this quadratic equation using the table above.

d Use the graph to estimate the area of the patio when the length is 0.75 m.

e Use the graph to estimate the area of the patio when the length is 2.75 m.

f What is the maximum area of the patio?

g What is the patio length in order to have maximum area?

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ISBN 978-1-108-44806-2

Cambridge University Press